

Math 123: Improper Integrals

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Outline

1 Improper Integrals

Improper integrals

Definite integrals $\int_a^b f(x)dx$ are required to have

- finite domain of integration $[a, b]$
- finite integrand (i.e. $f(x) < \pm\infty$)

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Improper integrals

- 1 Infinite domains of integration
- 2 Integrands with vertical asymptotes (i.e. with infinite discontinuity)

Infinite limits of integration

Definition

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{s \rightarrow -\infty} \int_s^a f(x) dx + \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

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If as a limit the improper integral is finite we say the integral **converges**, while if the limit is infinite or does not exist, we say the integral **diverges**.

Convergence is good (means we can do the integral); divergence is bad (means we can't do the integral).

Example 1

Find

$$\int_0^{\infty} e^{-x} dx.$$

(if it even converges)

Example 2

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

(if it even converges)

Example 3, the p -test

The integral

$$\int_1^{\infty} \frac{1}{x^p} dx$$

- ① Converges if $p > 1$;
- ② Diverges if $p \leq 1$.

Example 4

Find

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If $f(x)$ has a discontinuity at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^+} \int_a^t f(x) dx$$

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$$\int_0^3 \frac{1}{(x-1)^{2/3}} dx = \left[3(x-1)^{1/3} \right]_0^3,$$

but this is not okay!

Tests for convergence and divergence

The gist:

- 1 If you're smaller than something that converges, then you converge.
- 2 If you're bigger than something that diverges, then you diverge.

Theorem

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

- 1 $\int_a^\infty f(x) dx$ converges if $\int_a^\infty g(x) dx$ converges.
- 2 $\int_a^\infty g(x) dx$ diverges if $\int_a^\infty f(x) dx$ diverges.

Example 6

Which of the following integrals converge?

$$(a) \int_1^{\infty} e^{-x^2} dx, \quad (b) \int_1^{\infty} \frac{\sin^2(x)}{x^2} dx.$$