Math 123: Trig Substitution and Partial Fractions

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Thursday September 5, 2013

Outline

1. Trig Substitution

2. Partial Fraction Expansion

Trig Substitution

For integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{x^2 + a^2}$ where a is a constant, we can often integrate by constructing a right triangle with one of these values as a side length.

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Example: Find $\int \frac{1}{\sqrt{x^2-4}}$
Example: Find $\int \frac{x^2}{\sqrt{9-25x^2}}$

Making Hard Integrals Easy

Here is an easy integral

$$\int \frac{1}{x-3} + \frac{2}{x-4} dx$$

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Key Idea: The method of partial fractions expresses rational functions $\frac{p(x)}{q(x)}$ as the sum of simple fractions that we can integrate.

Steps of Partial Fraction Expansion

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Step 2: Factor the denominator (sometimes this is quite hard)
Example: Completely factor $x^3 - x$.

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In this case we let

$$\frac{p(x)}{q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_1)} + ... + \frac{A_k}{(a_kx + b_k)}$$

and we solve algebraically for $A_1, A_2, ..., A_k$.
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Example Find $\int \frac{3x-10}{x^2-7x+12}dx$

Example Find $\int \frac{1}{x^3-x}dx$

When the Denominator has Repeated Linear Factors

Step 3: Case 2: $q(x)$ is the product of linear factors, some of which are repeated

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$$\frac{x^2-3x+4}{(x-2)^2(x+3)^3} = \frac{A_1}{(x-2)} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x+3)} + \frac{A_4}{(x+3)^2} + \frac{A_5}{(x+3)^3}$$

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Key Idea: If $(x-a)$ appears n times in the factorization, we need n fractions on the right, one for each power.

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When the Denominator has all Irreducible Quadratic Factors

Irreducible Quadratics can not be factored into linear factors (over the reals).

$$x^2 + 1, 2x^2 - 2x + 4, -3x^2 + x - 1$$

Question: How do we find a partial fraction expansion if the denominator contains irreducible quadratics

Key Idea: For each irreducible quadratic factor we add one fraction to the right with numerator $Ax + B$.

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