

1. (10 pts) Evaluate the following limit. Carefully justify your answer.

$$\lim_{n \rightarrow \infty} \frac{\sin(n) \ln(n)}{n}$$

Use the squeeze thm.

$$-1 \leq \sin(n) \leq 1$$

$$\frac{-\ln(n)}{n} \leq \frac{\sin(n) \ln(n)}{n} \leq \frac{\ln(n)}{n}$$

Examine $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n}$ ← this is an indeterminate form of $\frac{\infty}{\infty}$.

Use L'Hospital's

$$\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\lim_{n \rightarrow \infty} \frac{-\ln(n)}{n} = -\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = -0 = 0.$$

Since $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = \lim_{n \rightarrow \infty} \frac{-\ln(n)}{n} = 0$ and

$\frac{-\ln(n)}{n} \leq \frac{\sin(n) \ln(n)}{n} \leq \frac{\ln(n)}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{\sin(n) \ln(n)}{n} = 0.$$

2. (10 pts) Evaluate the following series

$$\sum_{n=0}^{\infty} \frac{3^{n-1} + 4^{n+1} + 1}{5^n}$$

Recall: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{3^{n-1} + 4^{n+1} + 1}{5^n} &= \sum_{n=0}^{\infty} \frac{3^{n-1}}{5^n} + \frac{4^{n+1}}{5^n} + \frac{1}{5^n} \\ &= \sum_{n=0}^{\infty} \frac{3^{n+1}}{5^n} + \sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} + \sum_{n=0}^{\infty} \frac{1}{5^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} 4 \left(\frac{4}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n \\ &= \frac{1}{3} \left(\frac{1}{1 - \frac{3}{5}} \right) + 4 \left(\frac{1}{1 - \frac{4}{5}} \right) + \frac{1}{1 - \frac{1}{5}} \\ &= \frac{1}{3} \left(\frac{5}{2} \right) + 4(5) + \frac{5}{4} \\ &= \frac{10 + 240 + 15}{12} \\ &= \boxed{\frac{265}{12}} \end{aligned}$$

3. (10 pts) Find all values of k for which the following series converges. Carefully justify your answer.

$$\sum_{n=1}^{\infty} \frac{n+1}{kn^3 + n^2 + n + 1}$$

Suppose $k=0$. Use limit Comparison

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test.

Example $\lim_{n \rightarrow \infty} \frac{\frac{n+1}{n^2+n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2+n}{n^2+n+1}$

Since degree of top equals degree of bottom
 $= \frac{1}{1} = 1$

Hence, by the limit comparison test the series diverges when $k=0$.

Suppose $k \neq 0$. Use limit comparison test

We know $\sum_{n=1}^{\infty} \frac{1}{kn^2}$ converges by p-series test.

Evaluate $\lim_{n \rightarrow \infty} \frac{\frac{n+1}{kn^3+n^2+n+1}}{\frac{1}{kn^2}} = \lim_{n \rightarrow \infty} \frac{kn^3+kn^2}{kn^3+n^2+n+1}$

Since deg. of top = deg. of bottom
 $= \frac{k}{k} = 1$.

Hence, by the limit comparison test, the series converges when $k \neq 0$.

4. (10 pts) Determine if the following series converges or diverges. Carefully justify your answer.

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{\frac{1}{100}n^2}$$

Use the root test

$$\begin{aligned} \text{Examine } \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(1 + \frac{1}{n}\right)^{\frac{n^2}{100}} \right|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(1 + \frac{1}{n}\right)^{\frac{n^2}{100}}} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{\frac{n}{100}} \\ &= \left(\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right)^{\frac{1}{100}} \\ &= e^{\frac{1}{100}} > 1 \end{aligned}$$

Hence, by the ~~lim~~ root test,
 $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{\frac{n^2}{100}}$ diverges.

5. Show that the following series converges conditionally. Carefully justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

Step 1: Show $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln(n)}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges

use integral test: Examine $\int_1^{\infty} \frac{\ln(x)}{x} dx$

let $u = \ln(x)$ $du = \frac{1}{x} dx$

$$\int_a^b u du = \frac{u^2}{2} \Big|_a^b$$

$$= \lim_{a \rightarrow \infty} \frac{(\ln(x))^2}{2} \Big|_1^a$$

$$= \lim_{a \rightarrow \infty} \frac{(\ln(a))^2}{2} - 0 = \infty$$

Hence $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by the integral test.

Step 2: Show $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges

use Alternating Series test. with $b_n = \frac{\ln(n)}{n}$

① $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} \stackrel{\text{L'H.}}{=} \lim_{n \rightarrow \infty} \frac{1/n}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

② Show b_n is decreasing: Let $f(x) = \frac{\ln(x)}{x}$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$$

So, $f'(x) \leq 0$ for $x \geq e$. Thus, b_n is decreasing

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the alt. Series test.

By steps 1 and 2 $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges conditionally.

6. (10 pts) Find the interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

Step 1: Find the radius of convergence.

$$R = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n!)^2}{(2n)!}}{\frac{((n+1)!)^2}{(2n+2)!}} \right| = \lim_{n \rightarrow \infty} \frac{(n!)^2 (2n+2)!}{(2n)! ((n+1)!)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

Since deg. of top = deg. of bottom
 $= \frac{4}{1} = \boxed{4}$

Step 2: Determine convergence when $x = 4, -4$.

$x = 4$: Examine $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$ use the test for divergence.

Look at $\lim_{n \rightarrow \infty} \frac{(n!)^2 4^n}{2n!} = \lim_{n \rightarrow \infty} \frac{n \cdot n \cdot (n-1)(n-1)(n-2)(n-2) \dots 5 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 4^n}{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5) \dots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$$= \lim_{n \rightarrow \infty} \frac{(2n)(2n)(2n-2)(2n-2)(2n-4)(2n-4) \dots 6 \cdot 6 \cdot 4 \cdot 4 \cdot 2 \cdot 2}{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5) \dots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \lim_{n \rightarrow \infty} \frac{2n}{2n-1} \cdot \frac{2n-2}{2n-3} \cdot \frac{2n-4}{2n-5} \dots \frac{6}{5} \cdot \frac{4}{3} \cdot \frac{2}{1}$$

each of these is bigger than 1, so ≥ 1

Hence $\lim_{n \rightarrow \infty} \frac{(n!)^2 4^n}{(2n)!} \geq 1 \neq 0$. Thus, $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$ diverges.

The same argument shows $\sum_{n=1}^{\infty} \frac{(n!)^2 (-4)^n}{(2n)!}$ diverges.

Thus, the interval of convergence is $\boxed{(-4, 4)}$