Math 123: Approximate Integration II

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Outline

Review From Last Time

$\int_a^b f(x)dx$ is approximated by each of the following

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x$$

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1})\Delta x$$

$$M_n = f(\frac{1}{2}(x_0 + x_1))\Delta x + f(\frac{1}{2}(x_1 + x_2))\Delta x + ... + f(\frac{1}{2}(x_{n-1} + x_n))\Delta x$$

Approximating by Trapezoids

Recall that the area of a trapezoid with parallel sides of length a and b and of height h is

$$A = \frac{1}{2}(a+b)h$$

When we use trapezoids to approximate the area under the curve, we get

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n))$$

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Example Show T_n is the average of L_n and R_n
Example Find T_4 for $\int_0^1 x^2 dx$

Simpson's Rule

First, find a useful formula for the area under the parabola $Ax^2 + Bx + C$ from $x = -h$ to $x = h$.
We can use this to show that

$$S_n = \frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots$$

$$\ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

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Improper integrals

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Improper integrals

1. Infinite domains of integration
2. Integrands with vertical asymptotes (i.e. with infinite discontinuity)

Infinite limits of integration

Definition

$$\int_a^{\infty} f(x)dx = lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = lim_{s\to-\infty} \int_s^a f(x)dx + lim_{t\to\infty} \int_a^t f(x)dx$$

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If as a limit the improper integral is finite we say the integral **converges**, while if the limit is infinite or does not exist, we say the integral **diverges**.

Convergence is good (means we can do the integral); divergence is bad (means we can't do the integral).

Example 1

Find

$$\int_0^\infty e^{-x}\ dx.$$

(if it even converges)

Example 2

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

(if it even converges)

Example 3, the p-test

The integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

1. Converges if $p > 1$;
2. Diverges if $p \leq 1$.

Example 4

Find

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If $f(x)$ has a discontinuity at b, then

$$\int_a^b f(x)dx = lim_{t \to b^-} \int_a^t f(x)dx$$

Example 5

$$\text{Find} \int_0^3 \frac{1}{(x-1)^{2/3}}\, dx, \quad \text{if it converges.}$$

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$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay!

Tests for convergence and divergence

The gist:

1. If you're smaller than something that converges, then you converge.
2. If you're bigger than something that diverges, then you diverge.

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Theorem

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\, dx$ *converges if* $\int_a^\infty g(x)\, dx$ *converges.*
2. $\int_a^\infty g(x)\, dx$ *diverges if* $\int_a^\infty f(x)\, dx$ *diverges.*

Example 6

Which of the following integrals converge?

$$(a)\ \int_1^\infty e^{-x^2}\ dx, \qquad (b)\ \int_1^\infty \frac{\sin^2(x)}{x^2}\ dx.$$

$$(c)\ \int_0^\infty \frac{tan^{-1}(x)}{2+e^x}$$