Math 123: Improper Integrals

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Tuesday Feb. 8, 2016

Outline

Improper integrals

Definite integrals $\int_a^b f(x)dx$ are required to have

- finite domain of integration $[a, b]$
- finite integrand (i.e. $f(x) < \pm\infty$)

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Improper integrals

1. Infinite domains of integration
2. Integrands with vertical asymptotes (i.e. with infinite discontinuity)

Infinite limits of integration

Definition

$$\int_a^{\infty} f(x)dx = lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = lim_{s\to-\infty} \int_s^a f(x)dx + lim_{t\to\infty} \int_a^t f(x)dx$$

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If as a limit the improper integral is finite we say the integral **converges**, while if the limit is infinite or does not exist, we say the integral **diverges**.

Convergence is good (means we can do the integral); divergence is bad (means we can't do the integral).

Example 1

Find

$$\int_0^\infty e^{-x}\, dx.$$

(if it even converges)

Example 2

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

(if it even converges)

Example 3, the p-test

The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

1. Converges if $p > 1$;
2. Diverges if $p \leq 1$.

Example 4

Find

$$\int_0^2 \frac{2x}{x^2 - 4}\, dx.$$

(if it converges)

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$$\int_0^2 \frac{2x}{x^2-4}\,dx.$$

(if it converges)

If $f(x)$ has a discontinuity at b, then

$$\int_a^b f(x)dx = lim_{t\to b^-}\int_a^t f(x)dx$$

Example 5

$$\text{Find} \int_0^3 \frac{1}{(x-1)^{2/3}}\, dx, \quad \text{if it converges.}$$

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Solution: We might think just to do

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Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay!

Tests for convergence and divergence

The gist:

1. If you're smaller than something that converges, then you converge.
2. If you're bigger than something that diverges, then you diverge.

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Theorem

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\, dx$ *converges if* $\int_a^\infty g(x)\, dx$ *converges.*
2. $\int_a^\infty g(x)\, dx$ *diverges if* $\int_a^\infty f(x)\, dx$ *diverges.*

Example 6

Which of the following integrals converge?

$$(a)\ \int_1^\infty e^{-x^2}\ dx, \qquad (b)\ \int_1^\infty \frac{\sin^2(x)}{x^2}\ dx.$$

$$(c)\ \int_0^\infty \frac{tan^{-1}(x)}{2+e^x}$$