Math 123: Calculus on Parametric Curves

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Outline

Parametric Curves

Curves in the plane that are not graphs of functions can often be represented by parametric curves.

Definition

A *parametric curve* in the xy-plane is given by $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$.

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Example: Find the parametric equation for the portion of the circle of radius R in the 3rd quadrant. Give the *terminal point* and the *initial point*.

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Example: Find the parametric equation for the unit circle in the plane.
Example: Find the parametric equation for the portion of the circle of radius R in the 3rd quadrant. Give the *terminal point* and the *initial point*.
Example: All graphs of functions in can be represented as a parametric curve.

Awesome Examples

Cycloid:

$$(t - sin(t), 1 - cos(t))$$

An Epitrochiod:

$$(11cos(t) - 6cos(\frac{11}{6}t), 11sin(t) - 6sin(\frac{11}{6}t))$$

Wolfram Breaker:

$$(sin(t) + \frac{1}{2}sin(5t) + \frac{1}{4}cos(2.3t), cos(t) + \frac{1}{2}cos(5t) + \frac{1}{4}sin(2.3t))$$

Derivatives of Parametric Curves

If y is a differentiable function of x and t and x is a differentiable function of t then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

when $\frac{dx}{dt} \neq 0$.

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Example: Derive this formula from the chain rule.
Example: Find the points on the cycloid with horizontal tangent lines.

Area and Parametric curves

Theorem

If the graph of $y = F(x)$ on $[a, b]$ is parameterized by $x = f(t)$ and $y = g(t)$ for $t \in [\alpha, \beta]$ then

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt$$

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Example: Find the area under one arch of the cycloid.

Arc length for parameterized curves

Theorem

Given a curve $C = (f(t), g(t))$ *with* $t \in [\alpha, \beta]$, *then the length of* C *is*

$$L = \int_{\alpha}^{\beta} \sqrt{\frac{df}{dt}^2 + \frac{dg}{dt}^2}\, dt$$

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Example: Derive this formula from the definition of integral and the Pythagorean theorem.

Example: Find the length of one arch of the cycloid.