1. Evaluate $\int \frac{\ln(x)}{\sqrt{x}}\,dx$

Use by-parts

$$u = \ln(x) \qquad v' = x^{-1/2}$$

$$u' = \frac{1}{x} \qquad v = \frac{1}{2}x^{1/2}$$

$$\int \frac{\ln(x)}{\sqrt{x}}\,dx = \frac{1}{2}x^{1/2}\ln(x) - \int \frac{1}{2}x^{1/2}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{2}x^{1/2}\ln(x) - \int \frac{1}{2}x^{-1/2}\,dx$$

$$= \boxed{\frac{1}{2}x^{1/2}\ln(x) - \frac{1}{4}x^{1/2} + C}$$

2. Evaluate $\displaystyle\int \frac{x^4}{(x+1)^2(x^2+1)}\,dx$

Step 1 $\deg(x^4) = 4 = 4 = \deg\left((x+1)^2(x^2+1)\right)$

So, must divide

$$(x+1)^2(x^2+1) = (x^2+2x+1)(x^2+1) = \cancel{x^3+2x^2+x+x^2+2x+1}$$

$$= \cancel{x^3+3x^2+3x+1}$$

$$= \cancel{x^4+2x^2+x^2+x^2+2x+1}$$

$$= \cancel{x^4+4x^2+2x+1}$$

$$= x^4+2x^3+x^2+x^2+2x+1$$

$$= x^4+2x^3+2x^2+2x+1$$

$$\begin{array}{r|l} & \quad 1 \\ x^4+2x^3+2x^2+2x+1 & x^4+0x^3+0x^2+0x+0 \\ & -(x^4+2x^3+2x^2+2x+1) \\ \hline & -2x^3-2x^2-2x-1 \end{array}$$

So, $\displaystyle\int \frac{x^4}{(x+1)^2(x^2+1)}\,dx = \int 1 + \frac{-2x^3-2x^2-2x-1}{(x+1)^2(x^2+1)}\,dx$

Step 2: $\displaystyle \frac{-2x^3-2x^2-2x-1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$

$$-2x^3-2x^2-2x-1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

if $x=-1$, then $2-2+2-1 = A\cdot 0 + B(2) + (Cx+D)0$

$$1 = 2B$$

$$\boxed{B = 1/2}$$

So, $-2x^3-2x^2-2x-1 = A(x^3+x^2+x^2+1)+\frac{1}{2}x^2+\frac{1}{2}+(Cx+D)(x^2+2x+1)$

$$-2x^3-2x^2-2x-1 = Ax^3+\cancel{Ax}\,2Ax^2+A+\frac{1}{2}x^2+\frac{1}{2}+Cx^3+2Cx^2+Cx+Dx^2+2Dx+D$$

$$-2x^3-2x^2-2x-1 = (A+C)x^3+(2A+\tfrac{1}{2}+2C+D)x^2+(C+2D)x+(A+\tfrac{1}{2}+D)$$

So, $\left.\begin{aligned} A+C &= -2 \\ C+2D &= -2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A-2D &= 0 \\ A+\tfrac{1}{2}+D &= -1 \end{aligned}\right\}$ $\begin{aligned} 2D+\tfrac{1}{2}+D &= -1 \\ 3D &= -\tfrac{3}{2} \\ \boxed{D = \tfrac{-1}{2}} \end{aligned}$

$A+\frac{1}{2}+D = -1$

Since $D=\frac{-1}{2}$ and $A-2D=0$

then $A+1=0$

$\boxed{A=-1}$

Since $A=-1$ and $A+C=-2$

then $-1+C=-2$ and $\boxed{C=-1}$

Step 3: Integrate

$$\int \frac{x^4}{(x+1)^2(x^2+1)}\,dx = \int 1+\frac{-1}{x+1}+\frac{1/2}{(x+1)^2}+\frac{-x-\frac{1}{2}}{x^2+1}\,dx$$

$$= x-\ln|x+1|+\left(\frac{-1}{2}\right)(x+1)^{-1}+\int\frac{-x}{x^2+1}\,dx+\int\frac{-\frac{1}{2}}{x^2+1}\,dx$$

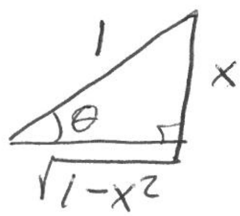
$u=x^2+1$

$du=2x\,dx$

$$= \boxed{x-\ln|x+1|+\left(-\frac{1}{2}\right)(x+1)^{-1}-\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\tan^{-1}(x)+C}$$

3. Exalude $\displaystyle\int \frac{1}{(1-x^2)^{3/2}}\,dx$

Use trig sub.



$$\frac{x}{1} = \sin\theta$$
$$dx = \cos\theta\, d\theta$$
$$\frac{\sqrt{1-x^2}}{1} = \cos\theta$$

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx = \int \frac{1}{(\cos\theta)^3}\cdot\cos\theta\, d\theta$$

$$= \int \sec^2\theta\, d\theta$$

$$= \tan\theta + C$$

$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

4. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{(\tan^{-1}(x))^2}{x^2+1}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{(\tan^{-1}(x))^2}{x^2+1}\,dx + \lim_{s\to\infty}\int_0^s \frac{(\tan^{-1}(x))^2}{x^2+1}\,dx$

use u-sub

let $u = \tan^{-1}(x)$

$$du = \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to-\infty}\int_{x=t}^{x=0} u^2\,du + \lim_{s\to\infty}\int_{x=0}^{x=s} u^2\,du$$

$$= \lim_{t\to\infty} \left.\frac{u^3}{3}\right|_{x=t}^{x=0} + \lim_{s\to\infty} \left.\frac{u^3}{3}\right|_{x=0}^{x=s}$$

$$= \lim_{t\to\infty} \left.\frac{(\tan^{-1}(x))^3}{3}\right|_t^0 + \lim_{s\to\infty} \left.\frac{(\tan^{-1}(x))^3}{3}\right|_0^s$$

$$= 0 - \frac{\left(\frac{-\pi}{2}\right)^3}{3} + \frac{\left(\frac{\pi}{2}\right)^3}{3} - 0$$

$$= \frac{\pi^3}{24} + \frac{\pi^3}{24} = \boxed{\frac{\pi^3}{12}}$$

5. Show that $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}}\,dx$ converges or diverges

Step 1:

$$\tan^{-1}(x) \geq 0 \text{ for } x \in (0,1)$$

$$x^{1/3} \geq 0 \text{ for } x \in (0,1)$$

Hence $\frac{\tan^{-1}(x)}{x^{1/3}} \geq 0$ for $x \in (0,1)$

Step 2: Since $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}}\,dx \sim \int_0^1 \frac{1}{x^{1/3}}\,dx$

we think it converges and we want to compare it to something like $\int_0^1 \frac{1}{x^{1/3}}\,dx$.

Step 3:

$$\tan^{-1}(x) \leq \frac{\pi}{2} \text{ (Remember}$$ 

)

So, $$\frac{\tan^{-1}(x)}{x^{1/3}} \leq \frac{\frac{\pi}{2}}{x^{1/3}}$$

Now we must evaluate

$$\int_0^1 \frac{\frac{\pi}{2}}{x^{1/3}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{\frac{\pi}{2}}{x^{1/3}} = \lim_{t \to 0^+} \frac{\pi}{2} \cdot \frac{3}{2} x^{2/3} \Big|_t^1$$

$$= \boxed{\frac{3\pi}{4}}$$

Since $0 \leq \frac{\tan^{-1}(x)}{x^{1/3}} \leq \frac{\frac{\pi}{2}}{x^{1/3}}$ and $\int_0^1 \frac{\frac{\pi}{2}}{x^{1/3}}\,dx$ converges, then $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}}\,dx$ converges.

6. Evaluate $\displaystyle\int_{-1}^{2} \frac{1}{x^{2/3}}\,dx$

Note $\dfrac{1}{x^{2/3}}$ has an infinite discontinuity at $x=0$, so, this is an improper integral.

$$\int_{-1}^{2} \frac{1}{x^{2/3}}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^{2/3}}\,dx + \lim_{s\to 0^+} \int_{s}^{2} \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{t\to 0^-} 3x^{1/3}\Big|_{-1}^{t} + \lim_{s\to 0^+} 3x^{1/3}\Big|_{s}^{2}$$

$$= \lim_{t\to 0^-} 3t^{1/3} - 3(-1)^{1/3} + \lim_{s\to 0^+} 3(2)^{1/3} - 3(s)^{1/3}$$

$$= \boxed{3 + 3\sqrt[3]{2}}$$

7. Evaluate $\int x \tan^{-1}(x)\,dx$

use by-parts

$$u = \tan^{-1}(x) \qquad v' = x$$

$$u' = \frac{1}{x^2+1} \qquad v = \frac{x^2}{2}$$

$$\int x \tan^{-1}(x)\,dx = \frac{x^2}{2}\tan^{-1}(x) - \int \frac{\frac{1}{2}x^2}{x^2+1}\,dx$$

use partial fractions

Step I $\deg(\frac{1}{2}x^2) = 2 = \deg(x^2+1)$

So, must divide

$$\begin{array}{r|l} x^2+0x+1 & x^2+0x+0 \\ & -(x^2+0x+1) \\ \hline & \quad -1 \end{array}$$

So, $$\frac{1}{2}\int \frac{x^2}{x^2+1}\,dx = \frac{1}{2}\int 1 + \frac{-1}{x^2+1}\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\tan^{-1}(x) + C$$

So, $$\int x \tan^{-1}(x)\,dx = \boxed{\frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C}$$