Knot Theory Day 22

<u>Outline</u>

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Announcement

- 1 page outline due Thursday
- HW due Thursday
- Sign ups for presentations May 1, 6, 8 (4 per day)

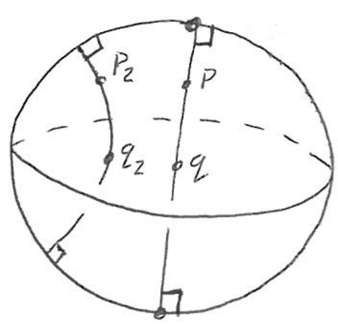
Recall: <u>Def</u>| A hyperbolic knot is a knot K s.t. $S^3 - K$ can be given a complete metric of constant curvature -1.

Last time we flew through hyperbolic 3-manifolds.

The simplest hyperbolic 3-manifold.

$\mathbb{H}^3 = \{(x,y,z) \mid x^2+y^2+y^2<1\}$ with a metric s.t. geodesics (shortest paths between points) are line segments and arcs of circles that meet $\partial \mathbb{H}^3$ in right angles.

<u>Ex</u>|



If ω is a geodesic between points p_1 and p_2 in $\mathbb{H}^3$, then the distance between p_1 and p_2 is $d(p_1, p_2) = \int_\omega \frac{2\,ds}{1-r^2}$

where we integrate wrt arc length in the euclidean metric and r is the distance to the orgin in euclidean space.

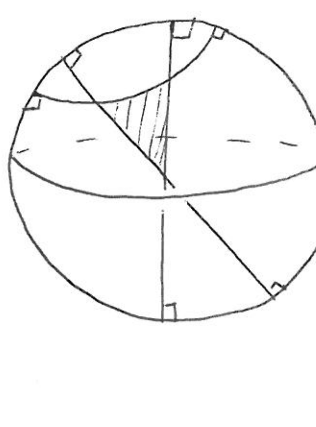
If $\omega = \langle x(t), y(t), z(t) \rangle$ is a geodesic in $\mathbb{H}^3$

$$d(\omega(t_0), \omega(t_1)) = \int_{t_0}^{t_1} \frac{2\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt}{1 - \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}^{\,2}}$$

For portions of geodesics near the orgin length in $\mathbb{H}^3$ is very close to ~~length in~~ twice the eucledian length. The length of geodesics becomes ~~length~~ longer, the closer they get to magnitude 1.

A <u>hyperbolic</u> <u>triangle</u> in $\mathbb{H}^3$ has edges consisting of subarcs of geodesics.

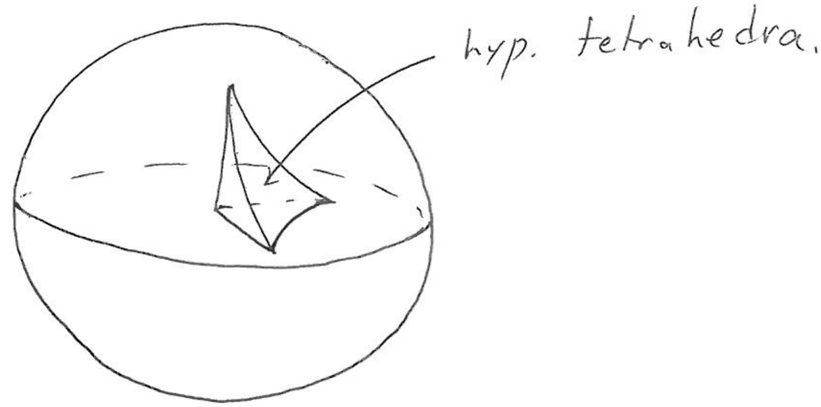
<u>Ex 1</u>



<u>Fact</u> Hyperbolic triangles always have a sum of angles less than π radians

<u>Fact</u> The area of a hyperbolic triangle is $\pi - (\alpha + \beta + \gamma)$ where α, β, γ and the angle measures in radians.

A <u>hyperbolic tetrahedra</u> has edges consisting of subarcs of geodesics and faces consisting of portions of planes and spheres that meet $\partial \mathbb{H}^3$ in right angles.



<u>Fact</u> | All hyperbolic tetrahedra have finite volume.

<u>Def</u> | A <u>hyperbolic 3-manifold</u> is a union of finitly many hyperbolic tetrahedra glue together allong faces via isometries (homeomorphisms with no metric distortion).

<u>Def</u> | A <u>hyperbolic knot</u> is a knot s.t. $S^3 - K$ is a hyperbolic 3-manifold.

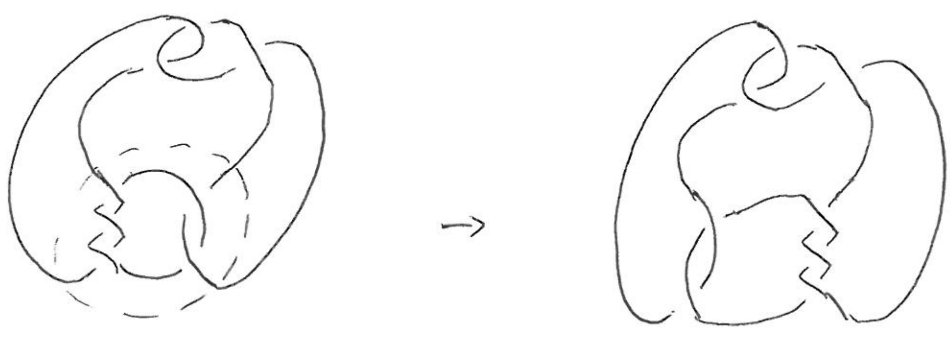
<u>Def</u> | Given a hyperbolic knot K, the sum of the volumes of all hyperbolic tetrahedra in $S^3 - K$ is the <u>volume of K</u>.

Thm | (Mostow rigidity) If $S^3 - K_1 \cong S^3 - K_2$ then K_1 and K_2 have the same volume.
(In fact, if $\pi_1(S^3 - K_1) \cong \pi_1(S^3 - K_2)$ then K_1 and K_2 have the same volume).

Vol () $\simeq 2.02988321\ldots$

Open Question | ① Find a knot with rational volume.
② Find a knot with irrational volume.

Let D be a knot diagram for a knot K.
Let B be a 3-ball in $\mathbb{R}^3$ that meets the plane of projection in a disk and K in ~~He~~ points equally destributed allong the equator. Let K' be the knot obtained by rotaiting B $180°$ about a line in theplane of projetion s.t. $K \cap \partial B^3$ gets mapped to $K \cap \partial B^3$. K' is called a mutant of K.

Ex


Fact | ~~Ex~~. If K' is a mutant of K then
$\mathrm{Vol}(K) = \mathrm{vol}(K')$.

Knot theory Day 23

Outline
- More theorems about 3-manifolds
- Alternating knots are not satellite.

Recall

Thm (Thurston) Let K be a knot in S^3, then K ~~has~~ falls into exactly one of the following categories
1) K is hyperbolic
2) K is satellite
3) K is a torus knot

Thm (Menasco) If K is a prime, alternating knot then K is not a satellite knot.

Our goal is to prove this theorem.

Cor (Menasco) If K is a prime, alternating knot that is not a $n/2$-torus knot then K is hyperbolic.

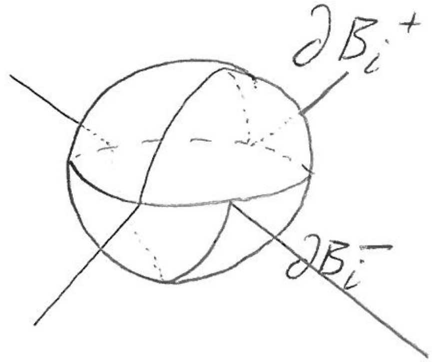
First we need some additional facts about Manifolds.

Thm | (Schönflies)
If γ is a polygonal (or smooth) curve embedded in S^2 then the closure of each of the two components of $S^2 - \gamma$ is a closed disk.

Thm | (Mazur) If S is a polygonal (or smooth) 2-sphere embedded in S^3, then the closure of each of the two components of $S^3 - S$ is a closed 3-ball.

Let K be a prime alternating knot in S^3 with diagram D and sphere of projection S.

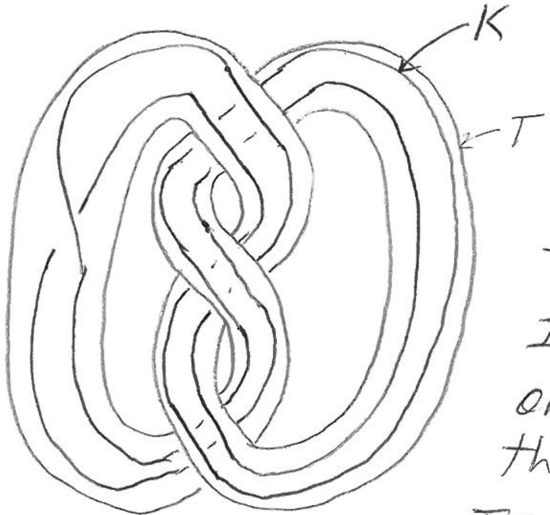
- Assume we have isotoped K to lie in an ε-nbh of P. Enumerat the crossings of D $1, \ldots, n$.
For each crossing of D incert a ~~2-sphere~~ 3-ball B_i s.t. $S \cap B_i$ is an equitorial disk in B_i and $S \cap \partial B_i$ divides ∂B_i into a northern hemisphere ∂B_i^+ and a southern hemisphere ∂B_i^-.
We can assume $K \subset S \cup (\bigcup_{i=1}^{n} \partial B_i)$ s.t. each over strand is contained in ∂B_i^+ and each under strand is contained in ∂B_i^-



Let $S^+ = (S - (\bigcup_{i=1}^{n} B_i)) \cup (\bigcup_{i=1}^{n} \partial B_i^+)$

$S^- = (S - (\bigcup_{i=1}^{n} B_i)) \cup (\bigcup_{i=1}^{n} \partial B_i^-))$

Suppose K is a satellite knot to derive a contradiction.



Let T be the companion torus in $S^3 - K$.

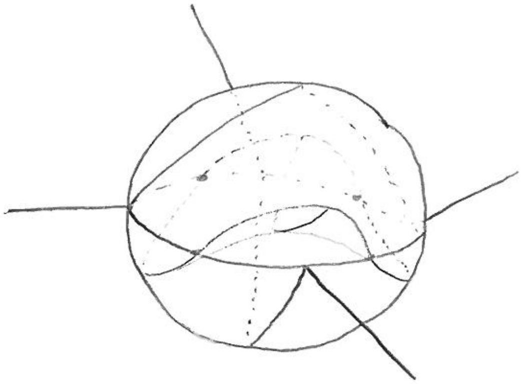
Claim) T is incompressible

If T has a comp. disk contained on the same side of T as K, then K is not satellite.

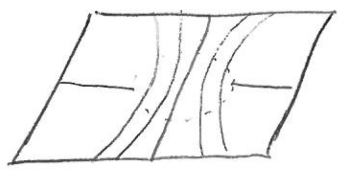
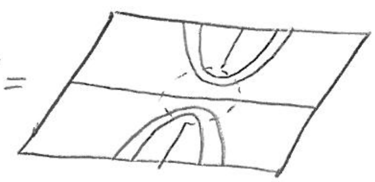
If T has a comp. disk contained on the oppisitte side as K, then K is not satellite (since the companion would be trivial).

Game plan: Investigate $T \cap S^+$ and $T \cap S^-$.

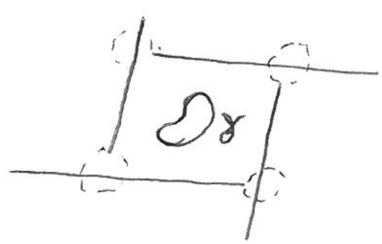
By choosing B small, we can assume $T \cap B_i$ is a collection of disks that meet B_i in a saddle.



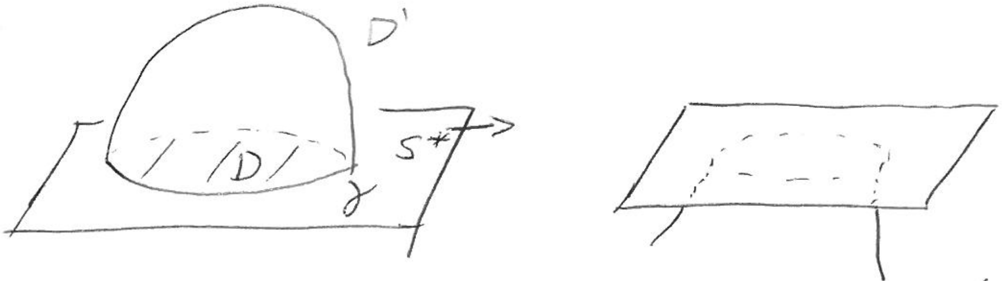
So if T meets B_i in two disks, then

$T \cap S^+ =$  $T \cap S^- =$ 

Claim ~~Lemma~~ We can assume No curve of $T \cap S^+$ is disjiont from $\cup \partial B_i^+$.
Suppose such a cure γ exists. Then γ is contained in a region of D.



Let γ' be an inner most curve of $T \cap S^+$ that bounds a disk D in S that is disjiont from $\cup \partial B_i^+$. Since T is incompressible γ bounds a disk D' in T. Since $D \cup D'$ bounds a 2-sphere in S^3



there is an isotopy of T that eliminates γ as a curve of intersection. $\square$

Assume that we have isotoped T to minimize the following lexigraphical ordering $\left(|T\cap(\bigcup_{i=1}^{n} B_i)|, |T\cap S^+| + |T\cap S^-|\right)$.