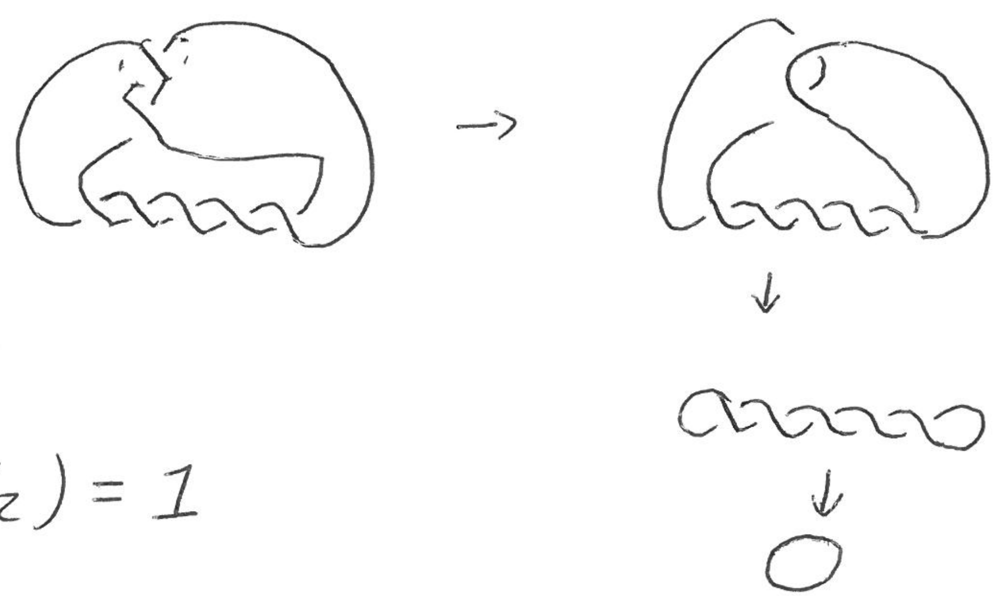
Knot Theory Day 7

- Outline
- unknotting number
- crossing number
- bridge number
- width

Unknotting Number

Def The unknotting number of a knot K is the minimal number of crossing changes nessiscery over all diagrams of K to change K to the unknot. (denoted $u(K)$).

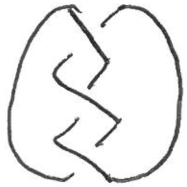
Ex 7_2



$u(7_2) = 1$

Def The crossing number of K is the minimal number of crossings in any diagram of K. (denoted $c(K)$)

Ex



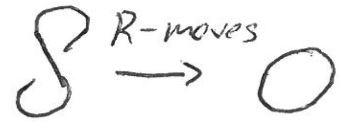
$c(\text{trefoil}) = 3$

Exercise Prove $c(\text{trefoil}) = 3$.

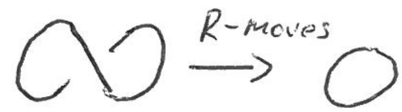
Claim If $c(K) = 1$, then K is the unknot.

Organize a, b, c, d in to all sets of two pairs

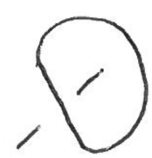
(a, b) (c, d)



(a, c) (b, d)



(a, d) (b, c)

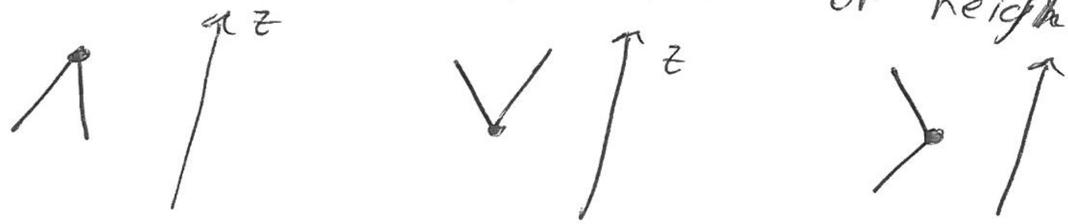


$\Rightarrow$ $\neq$ to one crossing.

Def
- Let $h : \mathbb{R}^3 \to \mathbb{R}$ s.t. $h(x, y, z) = z$.
- Let K be a knot in $\mathbb{R}^3$. We can move the vertices of K slightly so that no edge of K is parallel to the xy-plane

Hence all vertices of K are

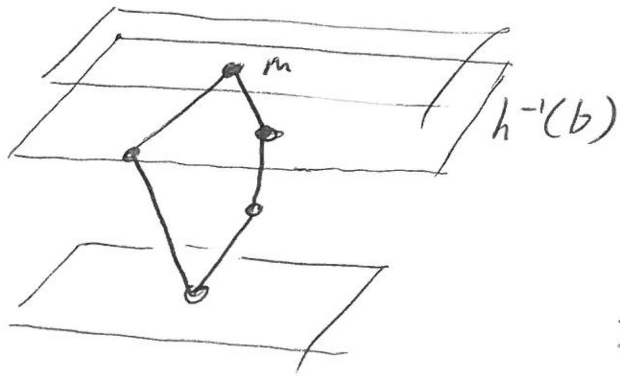
local maxima   local minima   or neither



The <u>bridge</u> <u>number</u> of K is the minimal number of local maxima over all knots equivalent to K. (denoted $\beta(K)$).

<u>Thm</u> If $\beta(K) = 1$, then K is an unknot.

<u>Idea</u> By Rolle's Thm, if any plane parallel to the xy-plane meets K in 3 or more points then K has at least 2 maxima. Hence, all planes meet K in 0, 1 or 2 points



Let b be the height of the first vertex of K that is not a maximum.

~~$h^{-1}(p) = \{v_1$~~

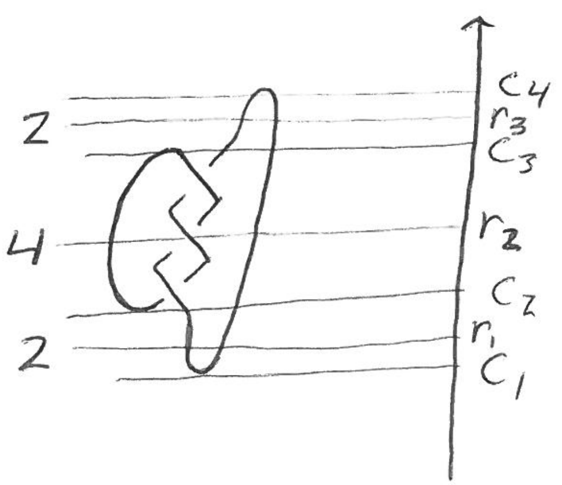
$h^{-1}(p) \cap K = \{v_1, v_2\}$

The triangle $\Delta v_1, v_2, m$ defines a E. D. of K that reduces its total height.

Repeat this process until K consists of exactly 4 vertices, then appeal to the homework problem.

<u>Ex</u>) $\beta(\text{trefoil}) = 2$

<u>Width</u>) Let $c_1 < c_2 < \cdots < c_{n+1}$ be the set of critical values of $h|_K$



Choose regular values r_i s.t. $c_i < r_i < c_{i+1}$ for $1 \le i \le n$.

$$\sum_{i=1}^{n} |K \cap h^{-1}(r_i)|.$$

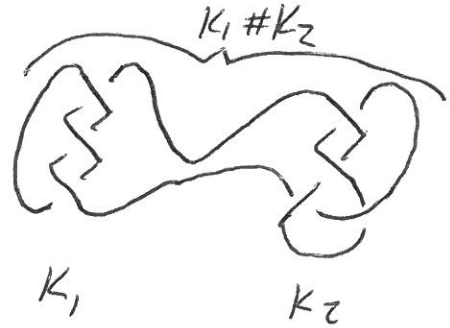
<u>Ex</u>) $w(\text{trefoil}) = 8$

The <u>width of a knot</u> is the minimum value over all knots equivalent to K. (denoted $w(K)$).

Theorems and open questions

Recall $K_1 \# K_2$ denotes the connected sum of K_1 and K_2

Ex



– If K cannot be decomposed as a ~~con~~ non-trival connected sum, we say K is prime.

Conj $u(K_1 \# K_2) = u(K_1) + u(K_2)$

Thm (Scharlemann) 85
If $u(K_1 \# K_2) = 1$, then K_1 or K_2 is the unknot.

Pf Very Hard!

Conj $c(K_1 \# K_2) = c(K_1) + c(K_2)$

Thm (Kauffman, Murasugi, Thistlethwaite) 88
If K_1 and K_2 are alternating, then $c(K_1 \# K_2) = c(K_1) + c(K_2)$

Def A knot is <u>alternating</u> if the crossings alternat under, over, under as you travel along <u>some</u> diagram of the knot.

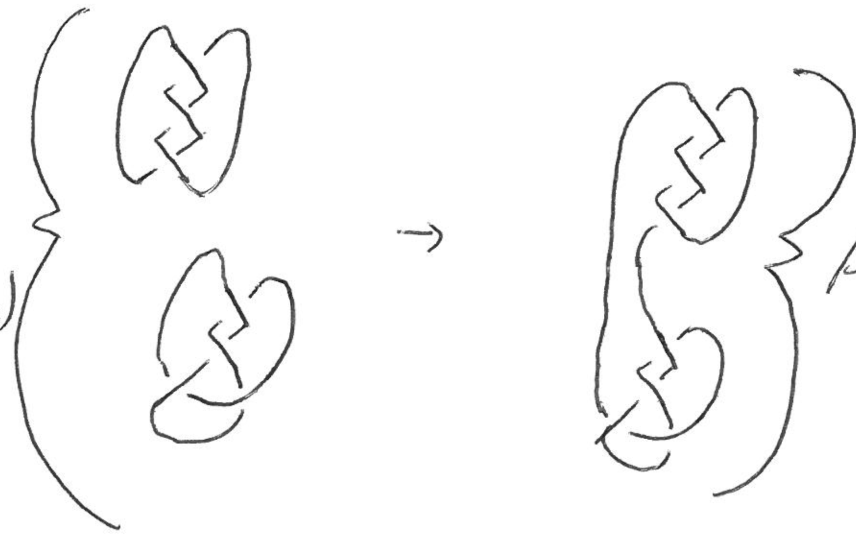
Thm (Schubert)

$$\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1$$

Pf Hard. See paper by Schultens.

Ex

$\beta(K_1 \sqcup K_2)$
$= \beta(K_1) + \beta(K_2)$



$\beta(K_1 \# K_2) \le \beta(K_1) + \beta(K_2)$

Conj $\omega(K_1 \# K_2) = \omega(K_1) + \omega(K_2) - 2$

False!

Thm (Blair, Tomova) There exist knots K_1 and K_2 s.t. $\omega(K_1 \# K_2) < \omega(K_1) + \omega(K_2) - 2$.

Knot theory Day 8

Outline

- Wrap-up classical invariants
- Tait Conjectures
- Kauffman bracket

From Last time

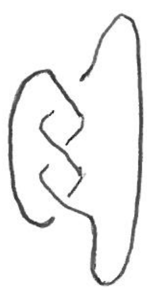
$u(K)$ is the unknotting number of K

$\beta(K)$ is the bridge number of K

$c(K)$ is the crossing number of K

$w(K)$ is the width of K

Ex



$u(\text{trefoil}) = 1$
$\beta(\text{trefoil}) = 2$
$c(\text{trefoil}) = 3$
$w(\text{trefoil}) = 8$

Thm (Scharlemann) $u(K_1 \# K_2) = u(K_1) + u(K_2)$ holds if $u(K_1 \# K_2) = 1$.

Thm (Schubert) $\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1$

Thm If $\beta(K) = 1$, then K is the unknot.

Def| A <u>diagram</u> of a knot is alternating iff traversing the knot diagram starting at any point and recording the crossings you meet as o (for over) or u (for under) produces the pattern ouou…ou or uouo…uo.

Ex|

ouououou

Def| A knot K is alternating if K has <u>some</u> diagram that is alternating

Thm| If $\beta(K)=2$, then K is alternating.

Thm| (Kauffman, Murasugi, Thistlethwaite)

If K_1 and K_2 are alternating $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

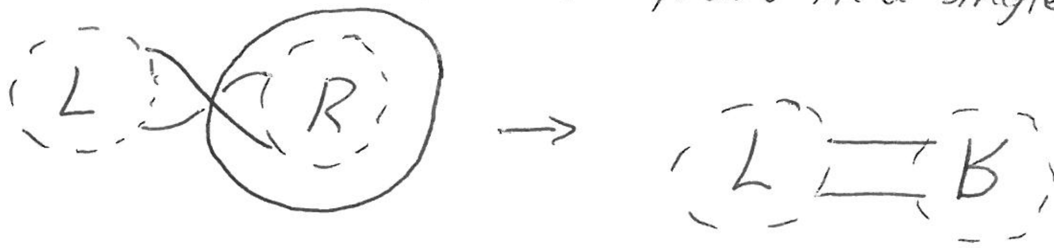
Tait Conjectures (Late 1800s)

1) Reduced alternating diagrams realize minimal crossing number.
2) Any two reduce alternating diagrams of a given knot have equal writhe

3) (flyping conjecture) Any two reduced alternating diagrams of a knot have the same number of crossings.

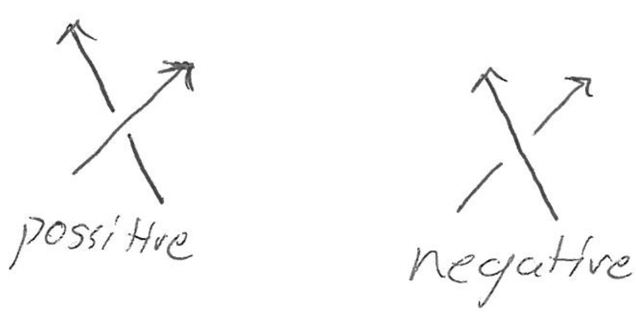
<u>Def</u> A diagram is <u>reduced</u> if it has no isthmuses

Isthmus: There exists a circle in the plane meeting the knot diagram in a single crossing.



<u>Def</u> An <u>orientation</u> on a knot is a choice of forward direction (~~choice o~~ consistant choice of unit tangent vector)

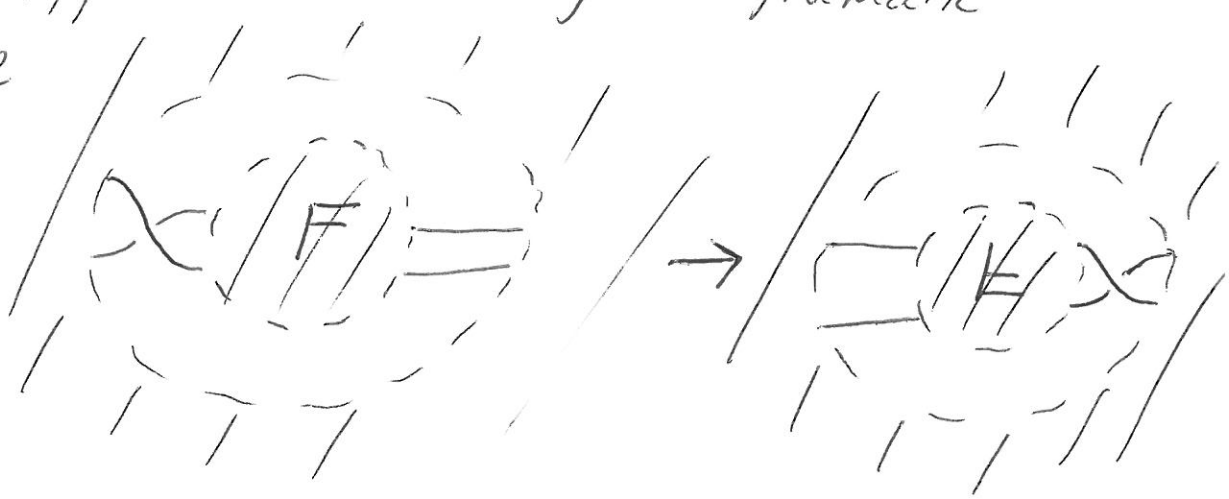
<u>Def</u> The <u>writhe</u> of ~~an~~ oriented knot diagram is the number of positive crossings minus the number of negative crossings.



Ex

$w(D) = 2 - 2 = 0$

Def A flype is the following diagramatic move



Thm Any two reduce alternating diagrams of a knot are related by a sequence of flypes.

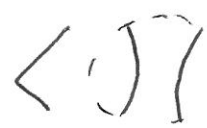
* flype is an old Scottish word meaning "to fold or turn back".

All of the tait conjectures were proved by using the Jones polynomial.

Kauffman bracket polynomial

<u>Def</u> The Kauffman bracket Polynomial of a link diagram D is a Laurent polynomial $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$, defined by the rules

(0) It is invariant under R0 moves

(1) $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

(skein relation)

(2) $\langle D \amalg U \rangle = (-A^2 - A^{-2}) \langle D \rangle$ where U is any crossing less diagram of the unknot.

(3) $\langle U \rangle = 1$ (this is a normalization).

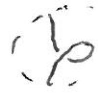
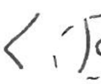
<u>Ex</u> $\langle$  $\rangle = A \langle \bigcirc \bigcirc \rangle + A^{-1} \langle$  $\rangle$

$$= A(-A^2 - A^{-2}) \langle \bigcirc \rangle + A^{-1} \langle \bigcirc \rangle$$

$$= -A^3 - A^{-1} + A^{-1} = -A^3$$

So, this is <u>not</u> a knot invariant.

__Example__ Suppose D is related to D^+ by a single $R1^+$ move (i.e.  $\xrightarrow{R1^+}$ )

$$\langle D^+ \rangle = \langle$$  $$\rangle = A \langle$$  $$\rangle + A^{-1} \langle$$  $$\rangle$$

$$= A(-A^2 - A^{-2}) \langle$$  $$\rangle + A^{-1} \langle$$  $$\rangle$$

$$= (A(-A^2 - A^{-2}) + A^{-1}) \langle D \rangle$$

$$= (-A^3) \langle D \rangle$$

__Claim__ The Kauffman bracket poly is invariant under RII and $RIII$ moves.

$$\langle$$  $$\rangle = A \langle$$  $$\rangle + A^{-1} \langle$$  $$\rangle$$

$$= A\Big(A \langle$$  $$\rangle + A^{-1} \langle$$  $$\rangle\Big)$$

$$+ A^{-1}\Big(A \langle$$  $$\rangle + A^{-1} \langle$$  $$\rangle\Big)$$

$$= A^2 \langle$$  $$\rangle + \langle$$  $$\rangle$$

$$- A^2 \langle$$  $$\rangle - A^{-2} \langle$$  $$\rangle + A^{-2} \langle$$  $$\rangle$$

$$= \langle$$  $$\rangle. \checkmark$$

Exercise Show the Kauffmann bracket poly is invariant under the R III move.

Correcting via writhe

Lemma The writhe of an oriented knot diagram is invariant under R II and R III, but changes under R I.

Pf