Math 500, Homework 4

Metric spaces and Connectedness

Due at start of class, Tuesday, **11/1**

Reading Read §20 – 21 and 26–28 of Munkres.

Exercises (to do on your own)

1. Prove that the collection of all ϵ-balls in a metric space actually forms a basis for a topology.
2. If (X, d) is a metric space, verify that $\overline{d}(x, y) = \min(d(x, y), 1)$ defines a metric on X.
3. Munkres §20, exercise 4.
4. Prove that the taxicab metric on $\mathbb{R}^n$ (see problem 2 below) defines a metric.
5. Prove the fact used in class that in the product space $X \times Y$, the subspace $X \times \{y\}$ is homeomorphic to X for any $y \in Y$.
6. Prove that $\mathbb{R}^\omega$ is not connected in the box topology. (Hint: recall $\mathbb{R}^\omega$ is the set of all sequences of real numbers. Now look at the subsets consisting of bounded and unbounded sequences.)
7. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n > 1$ (Hint: consider what happens if you delete a point from each space).
8. (a) If A is a connected subspace of X, is $\overline{A}$ necessarily connected?
   (b) If A is a path-connected subspace of X, is $\overline{A}$ necessarily path-connected?

Problems (to turn in)

1. Munkres §21, exercise 7 (recall $\mathbb{R}^X$ is the set of functions $X \to \mathbb{R}$).
2. Suppose that d and d' are metrics on a set X. d and d' are said to be *uniformly equivalent* if there exist positive real numbers a, b such that

$$ad(x, y) \leq d'(x, y) \leq bd(x, y)$$

   for all $x, y \in X$.

(a) Prove that if d and d' are uniformly equivalent, then they induce the same topology on X. (You may use results from Munkres.)

(b) Prove that on $\mathbb{R}^n$, the Euclidean metric d, the square metric ρ, the "taxicab metric"

$$\tau(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \ldots + |x_n - y_n|.$$

are all uniformly equivalent to each other.

3. Is $\mathbb{R}_\ell$ connected? Explain.

4. Munkres §24, exercise 3.

5. Given a topological space X, define a relation $\sim$ on X by setting $x \sim y$ if there exists a connected subspace of X containing both x and y.

   (a) Verify that $\sim$ is an equivalence relation. (See §3 for the definition of equivalence relation.)

   (b) A *component* of X is an equivalence class for $\sim$ (again, see §3). Prove that the components of X are connected, disjoint subspaces whose union is X.

   (c) Prove that any connected subspace of X is contained in some component.