Math 500, Homework 5

Compactness

Due in class, Tuesday, 11/7

Reading §26–28

Exercises (to do on your own)

1. Does every topological space have a finite cover?

2. Prove that the unit n-sphere S^n is compact.

3. (First, review $\liminf$ of a sequence of real numbers.) Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* if for every sequence of points $\{x_n\}$ converging to some $x \in X$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(Think of this as: if you approach a point x via a sequence $\{x_n\}$, the function f is allowed to "jump down", but not "jump up.")

   (a) Give an example of a function that is lower semi-continuous but not continuous.

   (b) Prove that if X is compact and $f : X \to \mathbb{R}$ is lower semi-continuous, then f achieves its minimum. Must f achieve its maximum?

4. Prove that $\mathbb{R}$ with the finite complement topology is compact.

Problems (to turn in)

1. Munkres §26, exercise 8. This is an example of a "closed graph theorem." (You may assume exercise §26.7, as the hint suggests. Recall that a closed map sends closed sets to closed sets.)

2. Munkres §28, exercise 7, parts a) and b) only.

(OVER)

3. Let $Z = \mathbb{R} \cup \{*\}$, where $\{*\}$ is a one-point set (that is not a subset of $\mathbb{R}$). Put a topology on Z using the basis consisting of all open intervals in $\mathbb{R}$, together with all sets of the form $(a, \infty) \cup \{*\} \cup (-\infty, -a)$ for $a > 0$. (Think of "gluing" the point $*$ in such a way as to join $-\infty$ and ∞.) Prove that S^1 is homeomorphic to Z by completing the following steps.

   (a) Recall that $x \times y \in \mathbb{R}^2$ belongs to S^1 iff $x^2 + y^2 = 1$. Define $f : S^1 \to Z$ by

   $$f(x, y) = \begin{cases} \frac{x}{1-y}, & \text{if } y \neq 1 \\ *, & \text{if } y = 1. \end{cases}$$

   Prove that f is bijective. (Hint: prove that if $y \neq 1$, then there exists a unique line in $\mathbb{R}^2$ containing the point 0×1 and the point $x \times y$. The value of $f(x, y)$ is where this line crosses the x-axis. f is called *stereographic projection.*)

   (b) Prove that f is continuous by showing the inverse image of any basic open set in Z is open in S^1. You may draw pictures to aid in your argument.

   (c) Use a trick from class to automatically conclude that f is a homeomorphism. Be sure to justify all your steps.

   Remark: An analogous argument shows that S^n is homeomorphic to $\mathbb{R}^n$ glued to a single "point at infinity" for any $n \geq 1$.