Topology Day 1

Outline

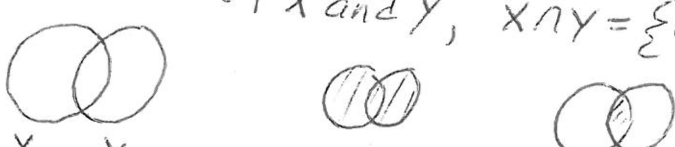
- Review of
  - sets and operations
  - functions
  - logic
- Define topological space
- Investigate simple examples

Sets and operations

- A <u>set</u> is a collection of objects called <u>elements</u>.
- "$x \in X$" means x is an object belonging to the set X.
- "$X \subset Y$" or "$X \subseteq Y$" or "$Y \supseteq X$" or "$Y \supset X$" means every element in X is also an element in Y.
- "$X = Y$" means $X \subset Y$ and $Y \subset X$.
- $\emptyset$ denotes the <u>empty set</u> (the set containing no elements)

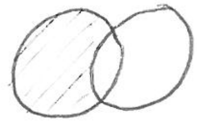
  ~~1) $X \cup \emptyset = X$~~

  ~~2) $X \cap \emptyset = \emptyset$~~
- If X and Y are sets
  - the union of X and Y, $X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}$
  - the intersection of X and Y, $X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}$



X Y $X \cup Y$ $X \cap Y$

- $X \cup \emptyset = X$ and $X \cap \emptyset = \emptyset$
- X minus Y, $X \setminus Y$ or $X - Y = \{a \mid a \in X \text{ and } a \notin Y\}$



$X - Y$

- The product of X and Y, $X \times Y = \{(a,b) \mid a \in X, b \in Y\}$

  Ex| $\mathbb{R}$ = the real line

  $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ = the cartesian plane.

Arbitrary unions and intersections

Let $\mathcal{A}$ denote a collection of sets.

$$\bigcup_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for some } A \in \mathcal{A}\}$$

$$\bigcap_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for every } A \in \mathcal{A}\}.$$

Example| $\mathcal{A} = \{(\frac{-1}{n}, \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ positive integers.

$$\bigcup_{A \in \mathcal{A}} A = (-1, 1) \qquad \bigcap_{A \in \mathcal{A}} A = \{0\}$$

Rules of set theory

1) Distributive laws: Let X, Y and Z be sets.
   a) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
   b) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

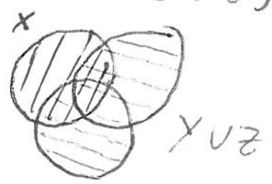
2) De Morgans Laws

a) $X-(Y\cup Z) = (X-Y)\cap(X-Z)$

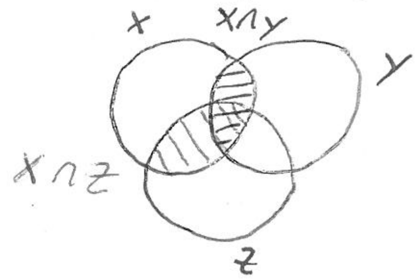
b) $X-(Y\cap Z) = (X-Y)\cup(X-Z)$

<u>Ex</u> 1a

$X\cap(Y\cup Z)$



$(X\cap Y)\cup(X\cap Z)$



Exercise: Prove 1 a) and b) and 2 a) and b).

<u>Functions</u>

A function $f: X\to Y$ associates to each $x\in X$ one element of Y, denoted $f(x)$.

- if $A\subset X$, you can form the restriction $f|_A : A\to Y$
- if $f: X\to Y$ and $g: Y\to Z$, define $(g\circ f): X\to Z$ s.t.

$$(g\circ f)(x) = g(f(x))$$

- f is <u>injective</u> or <u>one-to-one</u> if $x, x'\in X$ s.t. $x\neq x'$ implies $f(x)\neq f(x')$.
- f is <u>surjective</u> or <u>onto</u> if $\forall y\in Y$, $\exists x\in X$ s.t. $f(x)=y$.
- If f is both injective and surjective we say f is bijective.
- If $A\subset X$, the <u>image of A under f</u>, $f(A)=\{y\in Y \mid y=f(x) \text{ for some } x\in X\}$.

- If $B \subset Y$, the <u>preimage of B under f</u> is

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

<u>Rules for images and preimages</u> | If $A \subset X$, $B \subset Y$ and $f: X \to Y$

1) $A \subset f^{-1}(f(A))$ (equality holds if f is injective)
2) $f(f^{-1}(B)) \subset B$. (equality holds if f is surjective)

<u>Logic</u>

- If P and Q are statements "$P \Rightarrow Q$" means "P implies Q", or "If P is true, then Q is true".
- The <u>converse</u> of $P \Rightarrow Q$ is $Q \Rightarrow P$ ($Q \Rightarrow P$ <u>is not</u> logically equivalent to $P \Rightarrow Q$)
- $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$
- The <u>contrapositive</u> of $P \Rightarrow Q$ is not $P \Rightarrow$ not Q (not $P \Rightarrow$ not Q <u>is</u> logically equivalent to $P \Rightarrow Q$).

<u>Ex</u> | $P =$ "$x/2$ is an integer."
$Q =$ "x is an integer"
$P \Rightarrow Q$ is true, $Q \Rightarrow P$ is false

Contrapositive: If x is not an integer, then $x/2$ is not an integer.

<u>Ex</u> | $P =$ "$x < 0$ and $x > 0$."
$Q =$ "$x = 5$"
$P \Rightarrow Q$ is true (this is a <u>vacuous</u> truth)

Def | A topology on a set X is a collection of subsets $\mathcal{T}$ of X satisfying:

1) Any arbitrary union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
2) Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
3) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is a topological space.

Given a topological space $(X, \mathcal{T})$, $U \subset X$ is open if $U \in \mathcal{T}$.

Examples of topologies: Let X be any set.

$\mathcal{T} = \{\emptyset, X\}$ is the indiscrete topology.

$\mathcal{T} = \mathbb{P}(X)$ is the discrete topology.

(recall $\mathbb{P}(X)$ is the set of all subsets of X)

Topological Spaces

A motivating example: Let $X = \mathbb{R}$. From analysis we know $U \subset \mathbb{R}$ is <u>open</u> if ~~...~~ $\forall x \in U$ ~~...~~ $\exists\ (a,b)$ s.t. $x \in (a,b)$ and $(a,b) \subset U$.

Examples of open sets in $\mathbb{R}$.

1) $\mathbb{R}$
2) any open interval
3) $\emptyset$ (vacuously an open set).

$\{1\}$ is <u>not</u> open in $\mathbb{R}$.

Key properties of open sets in $\mathbb{R}$.

1) If $\mathcal{A}$ is any collection of open sets, $\bigcup_{A \in \mathcal{A}} A$ is open.
2) If $\mathcal{A}$ is any <u>finite</u> collection of open sets, $\bigcap_{A \in \mathcal{A}} A$ is open.

Proof of 1) Let $x \in \bigcup_{A \in \mathcal{A}} A$. By def. of arbitrary union, $x \in A_*$ for some $A_* \in \mathcal{A}$. By hypothesis, A_* is open. Hence, $\exists\ (a,b)$ s.t. $x \in (a,b) \subset A_*$. Since $A_* \subset \bigcup_{A \in \mathcal{A}} A$, then $x \in (a,b) \subset \bigcup_{A \in \mathcal{A}} A$. Thus, $\bigcup_{A \in \mathcal{A}} A$ is open. $\square$