Topology Day 11

Out line

- More Metric space
- Uniform metric and Uniform Convergence

Recall| A metric space (X,d) has a non-negative, symetric distance function $d: X \times X \to \mathbb{R}$ s.t. d satisfies the triangle inequality.

Def| (X,d) is <u>bounded</u> if $\exists M > 0$ s.t. $d(x,y) \leq M$ for all $x, y \in X$.

Note:| Since $(-\frac{\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$, boundedness is not a top. property.

Prop| Given a metric space (X,d) define

$$\bar{d}(x,y) = \min(d(x,y), 1)$$

Then $(X, \bar{d})$ is a bounded metric space and the metric topologies for (X,d) and $(X, \bar{d})$ agree.

Pf| - Claim: $\bar{d}$ is a metric

Pf| Exercise.

- $(X, \bar{d})$ is obviously bounded with $M = 1$.
- To show the metric topologies agree, show that

the set of $B_\varepsilon(x)$ with $\varepsilon < 1$ form a basis.

Hence, (X, d) and $(X, \bar{d})$ share a common (13.3 in book) basis and thus have the same topologies.

Motivating Question: How do we define a metric on $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$?

- can't use $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots}$ since sum might not converge.
- can't use $\sup_{n \geq 1} \{|x_n - y_n|\}$ is might be infinite.
- Trick define $\bar{d}(x, y) = \min(|x-y|, 1)$ on $\mathbb{R}$.

and define $\bar{\rho}(\vec{x}, \vec{y}) = \sup_{i \in \mathbb{Z}^+} \{\bar{d}(\vec{x}(i), \vec{y}(i))\}$.

<u>Def</u> $\bar{\rho}$ is the <u>uniform metric</u> and it induces the <u>uniform</u> <u>topology</u>.

<u>Thm</u> On $\mathbb{R}^\omega$

box top. is finer than uniform top. is finer than product top.

<u>Pf</u> (See Munkres 20.4 for rest) We will show box top. is finer than the uniform top. by showing basic open sets in the uniform top. are open in the

box top.

Let $B_\varepsilon(\vec{x})$ be a basis element in uni. top.

Let $\vec{y} \in B_\varepsilon(\vec{x})$ and let $\delta = \varepsilon - \bar{\rho}(\vec{x}, \vec{y}) > 0$.

Let $\mathcal{U} = \prod_{i=1}^{\infty} (\vec{y}(i) - \frac{1}{2}\delta, y(i) + 1/2\,\delta)$.

Since $\mathcal{U}$ is a basis element in box top., then $\mathcal{U}$ is open.

We need to show $\mathcal{U} \subset B_\varepsilon(\vec{x})$.

Let $\vec{z} \in \mathcal{U}$, then $|\vec{z}(i) - \vec{y}(i)| < \frac{1}{2}\delta$ for all $i \in \mathbb{Z}^+$.

Hence, $\bar{\rho}(\vec{z}, \vec{y}) \leq \frac{1}{2}\delta$.

So, $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z}) \leq \varepsilon - \delta + \frac{1}{2}\delta < \varepsilon$.

Hence, $\vec{z} \in B_\varepsilon(\vec{x})$ and $\mathcal{U} \subset B_\varepsilon(\vec{x})$

Hence, $B_\varepsilon(\vec{x})$ is open in box top. and the box top. is finer than the uniform top. $\square$

Ex| Consider the sequence
$$\vec{x}_1 = (1, 1, 1, \ldots)$$
$$\vec{x}_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots)$$
$$\vec{x}_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \ldots)$$

if $\mathbb{R}^\omega$

Claim $\vec{x}_n$ converges to $(0, 0, 0, \ldots)$ in uni. top.

Pf $\forall \varepsilon > 0$ $\exists M$ s.t. $\frac{1}{M} < \varepsilon$.
So, $\bar{\rho}(\vec{x}_n, \vec{0}) < \varepsilon$ for $n \geq M$.

Claim] $\vec{x}_n$ does not converge to $\vec{0}$ in box. top.

Pf] $\mathcal{U} = \prod_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i}\right)$ in a nbh of $\vec{0}$ that is disjiont from <u>every</u> element of $\{\vec{x}_n\}$.

Continuity in Metric Spaces

Let (X, d_X) and (Y, d_Y) be metric spaces

Def] $f: X \to Y$ is cts. iff $\forall x_0 \in X$, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$.

Note: A sequence $\{x_n\}$ in X converges to x iff $\{d(x, x_n)\}$ converges to 0.

Lemma] Let X be a top. space and $A \subset X$. If $\exists$ a sequence $\{a_n\}$ in A converging to $x \in X$, then $x \in \overline{A}$. (the converse holds if X is metrizable).

Pf] $\Rightarrow$] Say $\{a_n\} \to x$. Let $\mathcal{U}$ be a nbh of x. Then $\mathcal{U}$ contains all but finitly many of the a_n. Hence $\mathcal{U} \cap A \neq \emptyset$. Hence $x \in \overline{A}$.

$\Leftarrow$] Let d be a metric on X inducing the topology on X. Let $x \in \overline{A}$. For each n $B_{\frac{1}{n}}(x)$ is a nbh of x. Hence, $B_{\frac{1}{n}}(x) \cap A \ni a_n$ for each n. Then $\{a_n\}$ is a sequence in A and $d(a_n, x) < \frac{1}{n}$, so $\{a_n\} \to x$. $\square$

Another Chariterization of Continuity.

Prop| Let X, Y be top. spaces. If $f: X \to Y$ is cts.,
then for all convergent sequences $\{x_n\} \to x$, the
sequence $\{f(x_n)\}$ converges in Y to $f(x)$.
(Converse holds if X is metrizable.)

Pf| Exercise.

Uniform Convergence

Def| Given a set X and a metric space (Y, d), a
sequence of functions $f_n : X \to Y$ converepes
uniformly to $f: X \to Y$ if $\forall\, \varepsilon > 0\ \exists\, N > 0$ s.t.

$$d(f_n(x), f(x)) < \varepsilon \text{ for all } x \in X \text{ and all } n \geq N.$$

(Note N is independant of x).
The following is an important theorem from
analysis

Th| Given X a top. space and (Y, d) a metric space.
If $f_n : X \to Y$ is a sequence of continuous functions
converging uniformly to $f: X \to Y$, then f is continuous.

Pf| $\varepsilon/3$ argument from real analysis.