Topology Day 16

Outline

- Finite intersection property and compactness.
- Quotient space

From last time

Def A collect $\mathcal{C}$ of subsets of X has the finite intersection property if for every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$, then $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Thm X is compact iff every collection of closed subsets $\mathcal{C}$ with the finite intersection property also has the property $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Pf (Logic)

Let $\mathcal{U}$ be any collection of subsets of X and $\mathcal{C} = \{X - U \mid U \in \mathcal{U}\}$.

Note: ① Elements of $\mathcal{U}$ are open iff elements of $\mathcal{C}$ are closed

② $\mathcal{U}$ covers X iff $\bigcap_{C \in \mathcal{C}} C = \emptyset$

③ $\{U_1, \ldots, U_n\}$ covers X iff $C_1 \cap \cdots \cap C_n = \emptyset$.

The following is the contrapositive of the def. of compactness

X is compact iff "for any collection $\mathcal{U}$ of open sets, if no finite sub collection of $\mathcal{U}$ covers X, then $\mathcal{U}$ does not cover X."

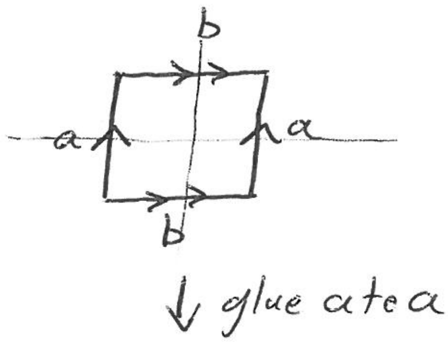
Equivalent to

X is compact iff "for any collection $\mathcal{C}$ of closed sets, if every finite subcollection has non empty intersection, then $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$" $\square$

Quotient Spaces

Motivation: We want a formal way of gluing together Top. spaces to get Top. spaces.

Ex



declare $(-1, y) \equiv (1, y)$ for $-1 \le y \le 1$

and

$(x, -1) \equiv (x, 1)$ for $-1 \le x \le 1$

Def| Let X and Y be top. spaces. Let $p: X \to Y$ be a surjective map. p is a quotient map if $V \subset Y$ is open iff $p^{-1}(V)$ is open in X.

Note: p is automatically continuous.

Ex| $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\pi_x((x,y)) = x$ is a quotient map

$$\pi_x^{-1}(A) = A \times \mathbb{R}$$

$A \times \mathbb{R}$ is open in $\mathbb{R}^2$ iff A is open in $\mathbb{R}$.

Lemma| Let X be a top space and Y be a set. Suppose $p: X \to Y$ is surjective. There exists a unique topology on Y s.t. p is a quotient map (call this the quotient top.).

Pf| Define $V \subset Y$ to be open ~~iff~~ if $p^{-1}(V)$ is open in X. Check that this defines a topology. p is automatically a quotient map. Check that this topology is unique.

Ex| Define a map $p: \mathbb{R} \to \{a, b, c\}$

s.t. $p(x) = \begin{cases} a & \text{if } x < 0 \\ b & \text{if } x = 0 \\ c & \text{if } x > 0 \end{cases}$

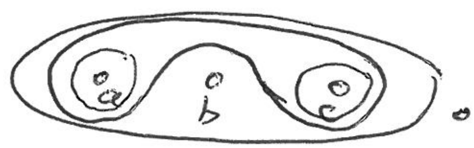
Want to construct the quotient topology on $\{a,b,c\}$

$p^{-1}(a) = (-\infty, 0)$

$p^{-1}(b) = \{0\}$

$p^{-1}(c) = (0, +\infty)$

So, we have the following topology



Note| Quotients of Hausdorff topologies need not be Hausdorff.

[Most examples of Quotient topologies come from equivalence relations.]

Recall $\sim$ is an equivalence relation on X if the following hold.

① $x \sim x \quad \forall x \in X$

② $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$

③ $x \sim y$ and $y \sim z \Rightarrow x \sim z$ for all $x, y, z \in X$.

An equivalence class of $x \in X$ under $\sim$ is

$$[x]_\sim = \{y \in X \mid y \sim x\}$$

Fact| Equivalence classes form a partition of X into disjiont subsets.

Def| Let X be a top. space with equivalence relation $\sim$. Let X^* be the set of equivalence classes of X under $\sim$. Let $p: X \to X^*$ be the map $p(x) = [x]_\sim$. Then we can equip X^* with the quotient topology.

$X^* = X/\sim$ is a quotient space of X

Ex| Let $X = \mathbb{R}$ and $x \sim y$ iff $x \equiv y \bmod 1$

$$X/\sim \;\cong\; S^1.$$

Ex| Let $X = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
and $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1.$

$$X/\sim \;\cong\; S^2.$$