Topology Day 18

Outline

- Quotient Spaces

Recall|

Lemma| If X is a top. space and Y is a set ~~then~~ and $p: X \to Y$ is surjective $\exists!$ topology on Y s.t. p is a quotient map. (i.e. $V \subset Y$ is open iff $p^{-1}(V)$ is open in X).

Def| A set X with a relation $\sim$ is an equivalence relation if

1. $x \sim x \quad \forall x \in X$
2. $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$
3. $x \sim y$ and $y \sim z \Rightarrow x \sim z \quad \forall x, y, z \in X$.

Let X be a top. space with an equivalence relation $\sim$.

An equivalence class is

$$[x]_\sim = \{ \cancel{x}y \in X \mid y \sim x \}.$$

Let $X^* = X/\sim$ be the set of equivalence classes of X under $\sim$.

Since $p : X \to X/\sim$ given by $p(x) = [x]_\sim$ is surjective, then by the lemma we can place a quotient topology on $X/\sim$.

Two representations of S^1 as the quotient of $\mathbb{R}$:

① $p : \mathbb{R} \to S^1$ $\qquad p(t) = e^{2\pi t i}$

$\uparrow$ a set

p is surjective ✓

Hence, S^1 recieves a unique quotient topology.

Check that this is equivalent to the standard top. on S!

② Let $\mathbb{R}$ be our top space

define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x \equiv y \mod 1$.

The set $\mathbb{R}/\sim$ recieves a quotient topology.

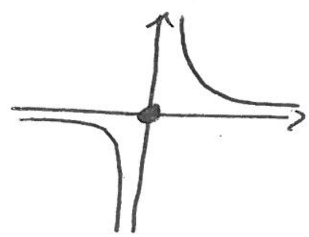
Show $\mathbb{R}/\sim$ is homeomorphic to S^1.

Be Careful!

__Fact__ If $p: X \to Y$ is a quotient map and $A \subset X$ is a subspace, $p|_A$ need not be a quotient map.

__Example__ Last time we showed
$\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\pi_1((x,y)) = x$
is a quotient map.

let $\mathbb{R}^2 \supset A = \{(0,0)\} \cup \{(x,y) \mid xy = 1\}$



$\pi_1|_A$ is not a quotient map since
$\{(0,0)\}$ is open in A but $\{(0,0)\} = (\pi_1|_A)^{-1}((0,0))$
and $\{(0,0)\}$ is not open in $\mathbb{R}$.

__Question__ For what topology on $\mathbb{R}$ is
$\pi_1|_A$ a quotient map?

__Ex__ A quotient of a Hausdorff space need not be Hausdorff.

Ex | If $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ are quotient maps, $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ need not be a quotient map. (Munkres Ex 7).

Lemma | If $p : X \to Y$ and $q : Y \to Z$ are quotient maps, then $q \circ p : X \to Z$ is a quotient map.

Pf | Since composition of surjective is surjective then $q \circ p$ is surjective.

Since composition of continuous is continuous then $q \circ p$ is continuous.

~~Examine~~ Suppose $(q \circ p)^{-1}(W)$ is open in X.
WTS W is open in Z. &

$$(q \circ p)^{-1}(W) = (p^{-1}) \circ ((q)^{-1}(W))$$

Since q is a quotient map $q^{-1}(W)$ is open in Y.
Since p is a quotient map $(p^{-1})(q^{-1}(W))$ is open in X. $\square$

More Examples

Ex Let $X = \mathbb{R}^{n+1} - \{\vec{0}\}$

Say $\vec{x} \sim \vec{y}$ if $\exists\ \lambda \in \mathbb{R}$ s.t. $\lambda \vec{x} = \vec{y}$.

$X/\sim$ is <u>real projective n-space</u> $\mathbb{RP}^n$ and can be thought of as the space of all lines through the orgin in $\mathbb{R}^{n+1}$.

Ex Let S^n have the subspace topology inherited from $\mathbb{R}^{n+1}$.

$\vec{x}, \vec{y} \in S^n$ say $\vec{x} \sim \vec{y}$ if $\vec{y} = -\vec{x}$

(we identify antipodal points).

$S^n/\sim \ \cong \mathbb{RP}^n$.

Ex $X = D^2 \times S^1$ (solid torus)

$Y = D^2 \times S^1$

~~Def~~ let $\varphi : \partial X \to \partial Y$ be a homeomorphism

Define $\sim$ on $X \amalg Y$ by

$x \sim y$ if $x \in \partial X$ and $y \in \partial Y$ s.t. $\varphi(x) = y$.

Depending on the choice of φ

$X \amalg Y / _n$ could be S^3

$S^2 \times S^1$

$\mathbb{RP}^3$

or any lens space.