Topology Day 20

OH
2-3 Fri
3-5 Monday

Outline

- Announcements
- Countability axioms

Recall|

Def| A top space X is ~~2nd~~ first-countable if given any $x \in X$ $\exists$ a countable collection of nbhs $\{U_i\}_{i=1}^{\infty}$ of x s.t. given any nbh V of x, $U_j \subset V$ for some j.

Def| A top space X is second-countable if X has a countable basis.

Note: Second-countable $\Rightarrow$ first-countable.
Let $\{U_i\}_{i=1}^{\infty}$ be the collection of all basis elements.

Ex| $\mathbb{R}_\ell$ is not second countable.
Let $\mathcal{B}$ be a basis of $\mathbb{R}_\ell$.
Examine $[x, x+1)$ and $[y, y+1)$ s.t. $x \neq y$.
Since $[x, x+1)$ and $[y, y+1)$ are open $\exists\, B_x, B_y \in \mathcal{B}$ s.t. $x \in B_x \subset [x, x+1)$ and $y \in B_y \subset [y, y+1)$
So $\inf B_x = x$ and $\inf B_y = y$ so $B_x \neq B_y$.

Def A subset A of a top space X is dense if $\overline{A} = X$.

Recall If $A \subset X$ then $x \in \overline{A}$ if $\forall$ nbh U of x $U \cap A \neq \emptyset$.

Def A subset A of a top space X is dense if given any open set $U \subset X$, $U \cap A \neq \emptyset$.

Def A top space X is separable if $\exists$ a subset $A \subset X$ s.t. $\overline{A} = X$ and A is countable.

Ex Our favorite separable space is $\mathbb{R}$.
$\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q}$ is dense.

Def A top space X is Lindelöf if any open cover of X has a ~~finite subcover.~~ countable subcover.

Thm If X is 2nd-countable, then X is Lindelöf and separable.

Pf First, WTS X is separable.
Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis for X.
Let $x_i \in B_i$ for each $B_i \in \mathcal{B}$.
Claim: $A = \{x_1, x_2, \ldots\}$ is dense in X.
Let U be an open set in X. By def of basis, $\exists B_j$ s.t. $B_j \subset U$. Hence $x_j \in U$.

Hence, if $x \neq y$ then $B_x \neq B_y$.

Thus, there are at least $\mathbb{R}$ distinct basis elements.

So, $\mathcal{B}$ is not countable.

<u>Prop</u> | A subspace of a first-countable space is first-countable, a countable product of a first countable space is first countable. (Same is true for 2nd-countable).

<u>Pf</u> | (We will prove the prop for 2nd-countable)

If $\mathcal{B}$ is a countable basis for X and $A \subset X$ is a subspace, then $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for A.

If $\mathcal{B}_i$ is a countable basis for X_i for $i = 1, 2, \ldots$

Then $\cancel{\mathcal{B}}$ $\mathcal{C} = \{\prod_{i=1}^{\infty} U_i \mid U_i = X_i$ for all but finitely many i and $U_i \in \mathcal{B}$ if $U_i \neq X_i\}$ is a basis for $\prod_{i=1}^{\infty} X_i$ under the product topology.

Claim: $\mathcal{C}$ is countable

$$\mathcal{C} = \{U_1 \times X_2 \times X_3 \times \cdots \mid U_1 \in \mathcal{B}_1\} \cup \{U_1 \times U_2 \times X_3 \times \cdots \mid U_1 \in \mathcal{B}_1, U_2 \in \mathcal{B}_2\} \cdots$$

So $\mathcal{C}$ is a countable union of countable sets. $\square$

Topology Day 21

Outline

- Countability axioms
- ~~Separation axioms~~ Manifolds.

Recall

X is

<u>first-countable</u> if $\forall x \in X$ $\exists \{U_i\}_{i \in \cancel{I}}$ a collection a nbh of x s.t. if V is a nbh of x then $U_j \subset V$ for some j.

<u>second-countable</u> if X has a countable basis.

<u>Lindelöf</u> if every cover of X has a countable subcover.

<u>Separable</u> if X contains a countable dense subset.

<u>Thm</u> If X is second-countable then X is separable and Lindelöf.

<u>Pf</u> (second-countable $\Rightarrow$ separable) last time.

WTS If X is second-countable, then X is Lindelöf.

Let $\mathcal{U}$ be a cover for X.

Let $\{B_i\}_{i \in \mathbb{Z}^+}$ be a basis for X.

Let $\mathcal{U}_i \subset \mathcal{U}$ be the subset ~~of $\mathcal{U}$~~ of sets that contain B_i.

Choose $U_i \in \mathcal{U}_i$ for each i for which $\mathcal{U}_i$ is non empty. Let $\mathcal{U}'$ be the collection of all ~~of~~ these U_i.

Claim: X is contained in the union of the elements in $\mathcal{U}'$.

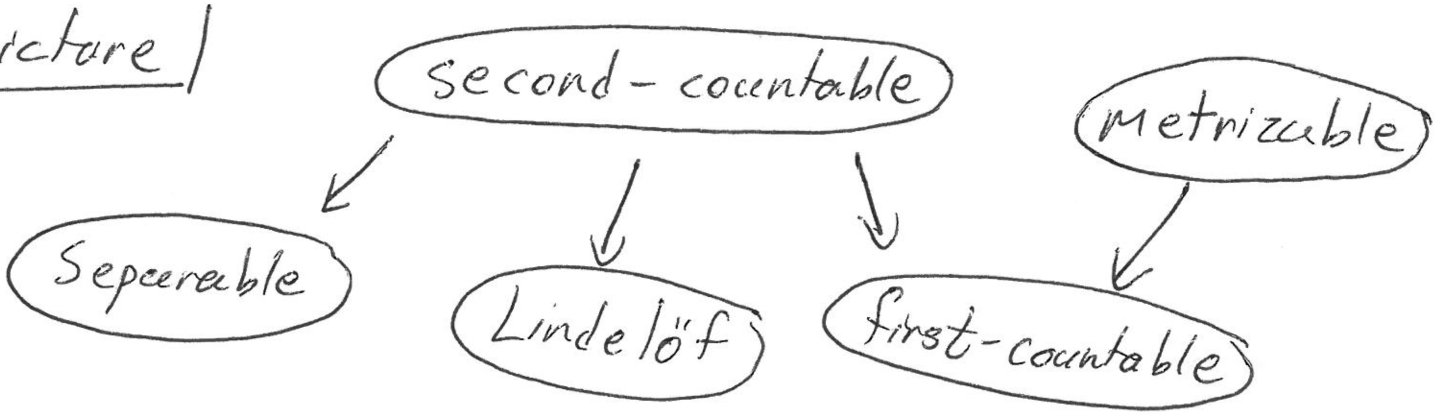
Pf: Let $x \in X$. Since $\mathcal{U}$ is a cover, $x \in U$ for some $U \in \mathcal{U}$. Since $\{B_i\}_{i \in \mathbb{Z}^+}$ is a basis, $x \in B_j \subset U$ for some j.

Hence $\mathcal{U}_j$ is non-empty and $x \in U_j$.

Thus X is contained in the union of the elements in $\mathcal{U}'$.

Since the basis is countable $\mathcal{U}'$ is countable and X is Lindelöf. $\square$

Picture



Examples

$\mathbb{R}_\ell$ is separable
($\mathbb{Q}$ intersects any basis element)

$\mathbb{R}_\ell$ is Lindelöf
(Munkres Ex. 3, § 30.)

Th^m ~~the~~ If X is metrizable, TFAE.

1) X is second-countable
2) X is seperable
3) X is Lindelöf.

Pf Showed 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) rest for HW.

Cor $\mathbb{R}_\ell$ is <u>not</u> metrizable since it is separable, but <u>not</u> ~~metri~~ 2nd countable.

Ex A product of Lindelöf spaces need not be Lindelöf.

Let $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$ with the product topology.

Let $L = \{(x,-x) \mid x \in \mathbb{R}_\ell\}$

Since $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is finer than $\mathbb{R}^2$ and L is closed in $\mathbb{R}^2$, then L is closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

<u>Claim</u> $(\mathbb{R}_\ell \times \mathbb{R}_\ell - L) \cup \{[a,a+1) \times [-a,-a+1) \mid a \in \mathbb{R}_\ell\}$ is an open cover of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Each element is open ✓.

If $(x,y) \in \mathbb{R}_\ell \times \mathbb{R}_\ell$ and $(x,y) \notin L$ then $(x,y) \in \mathbb{R}_\ell \times \mathbb{R}_\ell - L$.

If $(x,y) \in L$, then $y=-x$ and $(x,y) \in [x,x+1) \times [-x,-x+1)$.

So, this is a cover. □

<u>Claim</u> No strict subcover covers $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

If a strict subcover does not contain $\mathbb{R}_\ell^2 - L$, then it misses the points bellow $y=-x$.

If a sub cover does not contain $[a,a+1) \times [-a,-a+1)$, then it does not contain (a,a). □

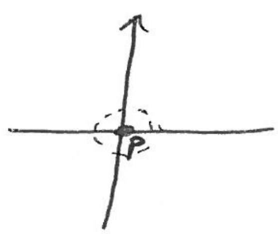
~~Th~~ Hence, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is <u>not</u> Lindelöf.

Def A top. space X is <u>Locally Euclidean</u> if every point p has a nbh U s.t. U is homeomorphic to an open subset of $\mathbb{R}^n$.

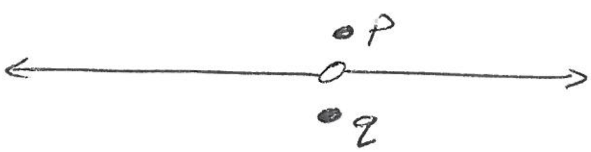
Ex • $\mathbb{R}^n$ is locally Euclidean ($U = \mathbb{R}$)

• S^1 is locally Euclidean

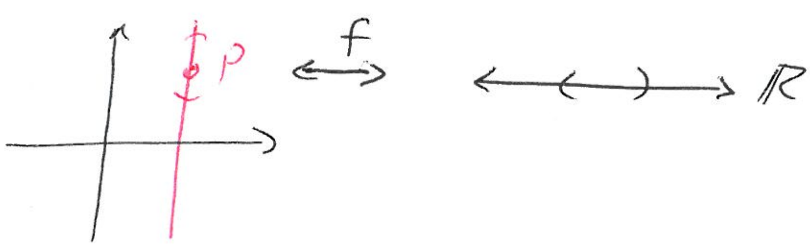
• The union of the x and y axis in $\mathbb{R}^2$, is <u>not</u> locally Euclidean



• The line with two orgins <u>is</u> locally Euclidean.



• $\mathbb{R} \times \mathbb{R}$ with dictionary order is locally Euclidean



Def An ~~manifold~~ n-manifold is a locally Euclidean, Hausdorff, second countable topological space.