Topology Day 22

Outline

- Separation axioms
  (T

– Announcements
A 50-39
B 38-29
C 28-20

Ave: 31
Median: 36

– Recall| A top. space X is T_1 if for any $x, y \in X$ s.t. $x \neq y$, there exists U_x a nbh of x and U_y a nbh of y s.t. $x \notin U_y$ and $y \notin U_x$.

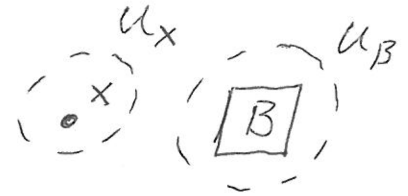
A top. space is T_2 (Hausdorff) if for any $x, y \in X$ s.t. $x \neq y$, there exists U_x a nbh of x and U_y a nbh of y s.t. $U_x \cap U_y = \emptyset$.

Note: ① X is T_2 implies X is T_1.

② X is T_1 iff all one point sets are closed.

Def| X is T_3 (regular) [terrible termonology.] if X is T_1 and for any $x \in X$ and $B \subset X$ s.t. B is closed (and does not contain x), there exists a nbh U_x of x and an open set U_B s.t. $B \subset U_B$ with $U_x \cap U_B = \emptyset$.

Pic|



Case 1: $B = (a,b)$ for $a<0<b$.

But $(a,b) \cap K \neq \emptyset$, so $U_0 \cap U_k \neq \emptyset$. $\divideontimes$.

Case 2: $B = (a,b) - K$ for $a<0<b$.

Let $k \in K$ s.t. $k<b$ and let $B_k \in \mathcal{B}$ s.t.

$k \in B_k \subset U_k$

Case 2a: $B_k = \cancel{(a,b)}\ (c,d)$

$(a,b)-K \cap (c,d) \neq \emptyset$ so $\divideontimes$

Case 2b: $B_k = (c,d) - K$.

Since $(a,b) \cap (c,d) \neq \emptyset$ and intervals are uncountable

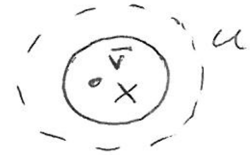
$(a,b) - K \cap (c,d) - K \neq \emptyset$. $\divideontimes$

Thus U_0 and U_k do not exist, so $\mathbb{R}_K$ is not T_3.

Lemma Assume X is T_1.

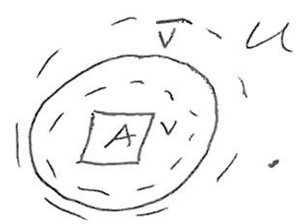
a) X is $T_3 \iff$ given $x \in X$ and a nbh U of x, $\exists$ a nbh V of x s.t. $\bar{V} \subset U$.

Pic.



b) X is $T_4 \iff$ given $A \subset X$ s.t. A is closed and U_A a nbh of A, there exists an open set V s.t. $A \subset V \subset \bar{V} \subset U$.

Pic



Mathematicians are bad at naming

Def | X is T_4 (normal) if X is T_1 and given any disjiont closed sets A and B, there exist open sets U_A and U_B s.t. $A \subset U_A$, $B \subset U_B$ and $U_A \cap U_B = \emptyset$.

Note | $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$.

– Previously we showed $\mathbb{R}$ with the finite complement topology is T_1, but not T_2.

Claim | $\mathbb{R}_K$ is T_2 but not T_3.

Pf | Recall $\mathbb{R}_K$ has basis $\mathcal{B} = \{(a,b) \mid a<b\} \cup \{(a,b) - K \mid a<b\}$ where $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$.

First, WTS $\mathbb{R}_K$ is T_2. Let $x, y \in \mathbb{R}_K$ s.t. $x = y$.

~~then~~ Assume $x < y$.
$$U_x = \left(x-1,\ x + \frac{y-x}{2}\right)$$
$$U_y = \left(y - \frac{y-x}{2},\ y+1\right)$$

$U_x \cap U_y = \emptyset$.

Second, WTS $\mathbb{R}_K$ is not T_3.

K is closed since $\mathbb{R} - K$ is open.

Let U_0 be a nbh of 0 and U_K be a nbh of K.

s.t. $U_0 \cap U_K = \emptyset$. We want to derive a contradiction.

$\exists\, B \in \mathcal{B}$ s.t. $x \in B \subset U_0$

a) $\Rightarrow|$ Suppose X is T_3. Let $x \in X$ and $\mathcal{U}$ be a nbh of x. Let $A = X - \mathcal{U}$, a closed set. Since $x \in \mathcal{U}$ then $\{x\} \cap A = \emptyset$. By def of T_3 $\exists$ $\mathcal{U}_x$ a nbh of x and $\mathcal{U}_A$ a nbh of A s.t. $\mathcal{U}_x \cap \mathcal{U}_A = \emptyset$.

Claim: $\overline{\mathcal{U}_x} \subset \mathcal{U}$.

<u>Pf</u>| Let $y \in \mathcal{U}_A$. Since $y \in \mathcal{U}_A$ and $\mathcal{U}_A \cap \mathcal{U}_x = \emptyset$ then $y \notin \overline{\mathcal{U}_x}$. Thus,

$$\overline{\mathcal{U}_x} \subset X - \mathcal{U}_A \subset X - A = \mathcal{U}. \quad \square$$

~~Hence X~~

$\Leftarrow|$ Let $x \in X$ and B be a closed set in X s.t. $x \notin B$. Since B is closed $\mathcal{U} = X - B$ is a nbh of x. By assumption $\exists$ V a nbh of x s.t. $x \in V$ and $\overline{V} \subset X - B$. Since $\overline{V}$ is closed then $X - \overline{V}$ is ~~an~~ a nbh of B.

Hence V is a nbh of x

$X - \overline{V}$ is a nbh of B

and $V \cap (X - \overline{V}) = \emptyset$. $\square$

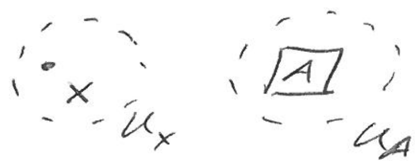
So, X is T_3.

b) Very similar argument.

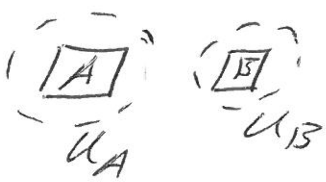
Topology Day 23

– More separation Axioms

Recall | X is T_3 if



X is T_4 if



Thm | A subspace of a T_3 space is T_3 and a product of T_3 spaces is T_3. (not true for T_4).

Pf | Suppose X is T_3. WTS $A \subset X$ with the subspace top is T_3.

First show A is T_1. Let $a \in A$, $\{a\}$ is closed in X since X is T_1. Hence, $\{a\} \cap A = \{a\}$ is closed in A and A is T_1.

Next, show A is T_3. Let $a \in A$ and let $B \subset A$ be a closed set s.t. $a \notin B$.

B = closure of B in $A = A \cap \overline{B}$

Closure in X.

Hence a is not contained in $\overline{B}$ and, since X is T_3, $\exists\, U_a$ a nbh of a and $U_{\overline{B}}$ a nbh of $\overline{B}$ s.t.

$\mathcal{U}_a$ and $\mathcal{U}_{\bar{\beta}}$ are disjiont.

By def of subspace top.

$\mathcal{U}_a \cap A$ is a nbh of a in A

$\mathcal{U}_{\bar{\beta}} \cap A$ is a nbh of $\bar{\beta}$ in A.

and $(\mathcal{U}_a \cap A) \cap (\mathcal{U}_{\bar{\beta}} \cap A) = \emptyset$.

So, A is T_3. $\square$

Next, Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of T_3 spaces and Let $\prod_{\alpha \in J} X_\alpha$ have the prod. top.

Since product of Hausdorff is Hausdorff, then $\prod_{\alpha \in J} X_\alpha$ is T_2 and, thus, T_1.

Let $\vec{x} = \{x_\alpha\}_{\alpha \in J}$ be a point in $\prod_{\alpha \in J} X_\alpha$.

Let $\mathcal{U}$ be a nbh of $\vec{x}$.

Let $\vec{x} \in \prod_{\alpha \in J} \mathcal{U}_\alpha \subset \mathcal{U}$ ($\mathcal{U}_\alpha \neq X_\alpha$ for only finitly many α).

Since X_α is T_3, then $\exists\, V_\alpha \subset X_\alpha$ s.t. $x_\alpha \in V_\alpha \subset \overline{V_\alpha} \subset \mathcal{U}_\alpha$ for each $\alpha \in J$. Let $V = \prod V_\alpha$.

$$\vec{x} \in \prod V_\alpha \subset \overline{\prod V_\alpha} \underset{\substack{\uparrow \\ \text{fact} \\ \text{about} \\ \text{product top.}}}{=} \prod \overline{V_\alpha} \subset \prod \mathcal{U}_\alpha \subset \mathcal{U}.$$

Hence, by previous lemma X is T_3. $\square$

Thm | If X is metrizable, then X is T_4 (normal).

Pf | Let A and B be closed sets in X.

Since $X-B$ is open $\forall a \in A$ $\exists$ ε_a s.t. $B_{\varepsilon_a}(a) \subset X-B$.

Since $X-A$ is open $\forall b \in B$ $\exists$ ε_b s.t. $B_{\varepsilon_b}(b) \subset X-A$.

Let $\mathcal{U}_A = \bigcup_{a \in A} B_{\frac{\varepsilon_a}{2}}(a)$ and $\mathcal{U}_B = \bigcup_{b \in B} B_{\frac{\varepsilon_b}{2}}(b)$.

Since they are the union of open sets $\mathcal{U}_A$ and $\mathcal{U}_B$ are open.

Claim: $\mathcal{U}_A \cap \mathcal{U}_B = \emptyset$.

Let $x \in \mathcal{U}_A \cap \mathcal{U}_B$.

Then there exists $p \in A$ s.t. $x \in B_{\frac{\varepsilon_p}{2}}(p)$ and there exists $q \in B$ s.t. $x \in B_{\frac{\varepsilon_q}{2}}(q)$

$$d(p,q) \leq d(p,x) + d(q,x) < \frac{\varepsilon_p}{2} + \frac{\varepsilon_q}{2}$$

WLOG assume $\varepsilon_p \leq \varepsilon_q$, then

$d(p,q) < \varepsilon_q$. This is a contradiction to how we chose ε_q.

Hence $\mathcal{U}_A \cap \mathcal{U}_B = \emptyset$ and X is T_4. $\square$

Thm | (32.3) A compact Hausdorff space is normal.

Pf | Exercise (uses ideas similar to the proof that compact subsets of Hausdorff spaces are closed)

Ex | $\mathbb{R}_\ell$ is T_4 (Recall: $\mathbb{R}_\ell$ is <u>not</u> metrizable)

Pf | $\mathbb{R}_\ell$ is T_2, so it is T_1.

Let $A, B \subset \mathbb{R}_\ell$ be disjiont closed subsets of $\mathbb{R}_\ell$.

Since $\mathbb{R}_\ell - A$ and $\mathbb{R}_\ell - B$ are open,

for any $a \in A$ ~~let $[a, x_a)$~~ chose $x_\alpha \in \mathbb{R}_\ell$ s.t.

$[a, x_\alpha) \cap B = \emptyset$.

for any $b \in B$ choose x_b s.t. $[b, x_b) \cap A = \emptyset$.

Let $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$.

Note: U is a nbh of A and V is a nbh of B.

Claim: $V \cap U = \emptyset$.

Let $p \in V \cap U$, then $p \in [a, x_\alpha) \cap [b, x_b)$

If $a \leq b$, then $b \in [a, x_a)$ $\divideontimes$.

If $a > b$, then $a \in [b, x_b)$ $\divideontimes$.

Hence $V \cap U = \emptyset$, and $\mathbb{R}_\ell$ is T_4. $\square$

Ex | $\mathbb{R}_\ell$ is T_4 but $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not T_4 (but is T_3).

Proof: Tricky.

Thm | If X is second countable and T_3, then X is T_4.

Pf | Let X be T_3 with countable basis $\mathcal{B}$ and let $A, B \subset X$ be closed disjiont subsets.

Given $x \in A$, $X - B$ is a nbh of x. By previous lemma. $\exists$ a nbh W_x of x s.t. $\overline{W_x} \subset X - B$. Since $\mathcal{B}$ is a basis $\exists$ $B_x \in \mathcal{B}$ s.t. $x \in B_x \subset W_x$.

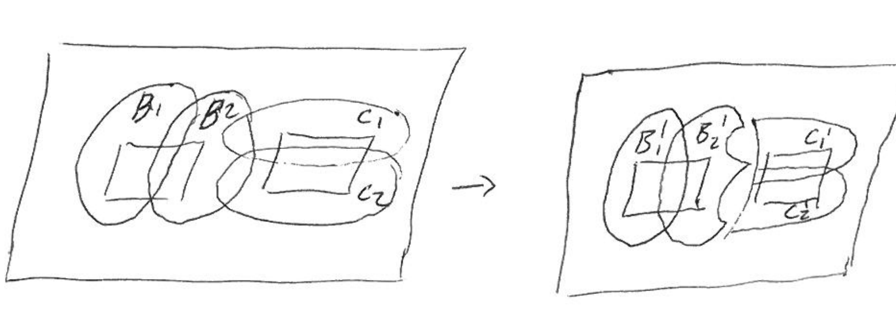
$\{B_x\}_{x \in A}$ is a cover of A and is countable. Hence rename this collection $\{B_i\}_{i=1}^{\infty}$. Construct ~~A~~ the coresponding cover $\{C_i\}_{i=1}^{\infty}$ for B.

For each n, let

$$B_n' = \underbrace{B_n - \left(\bigcup_{i=1}^{n} \overline{C_i}\right)}_{\text{open}} \qquad C_n' = \underbrace{C_n - \left(\bigcup_{i=1}^{n} \overline{B_i}\right)}_{\text{open.}}$$

Let $B' = \bigcup_{n=1}^{\infty} B_n'$ and $C' = \bigcup_{n=1}^{\infty} C_n'$

Pic |



Note: $A \subset B'$ and $B \subset C'$.

<u>Claim</u> $B' \cap C' = \emptyset$.

If $x \in B' \cap C'$ then $x \in B_i' \cap C_j'$.

WLOG Assume $i < j$.

Recall $C_j' = C_j - \left(\bigcup_{n=1}^{j} B_n \right)$, so $x \notin B_i$.

Since $B_i' \subset B_i$ then $*$ to $x \in B_i'$.

Hence, $B' \cap C' = \emptyset$. $\square$

Thus, X is T_4.

A pictoral summary of results

$$
\begin{tikzcd}
\text{metrizable} \arrow[r, Rightarrow] & T_4 \arrow[r, Rightarrow] & T_3 \arrow[r, Rightarrow] \arrow[l, bend left, "\text{2nd-countable}"] \arrow[ll, bend left=60, "\text{Urysohn Thm}"] & T_2 \arrow[r, Rightarrow] \arrow[ll, bend right, "\text{compact}"'] & T_1
\end{tikzcd}
$$

<u>Thm</u> (Urysohn Metrization thm)
Every regular space X with a countable basis is metrizable.