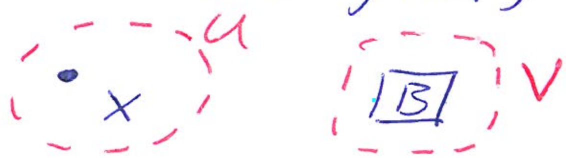
Topology Day 24

Outline
- List of Separation axioms
- Outline of Urysohn lemma & Urysohn Metrization Thm.

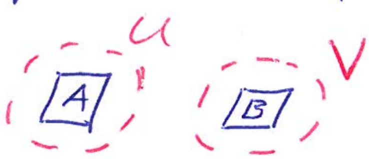
- Announcements
  - New HW up.

Recall:

A top. space is T_3 (regular) if



A top. space is T_4 (normal) if



Thm A compact Hausdorff space is normal

Pf Exercise (Similar to the proof that a compact subset of a Hausdorff space is closed).

Thm If X is 2nd-countable and T_3, then X is T_4.

(this is needed in the proof of the Urysohn Metrization theorem).

Pf] Let X be T_3 with countable basis $\mathcal{B}$. Let A and B be disjiont

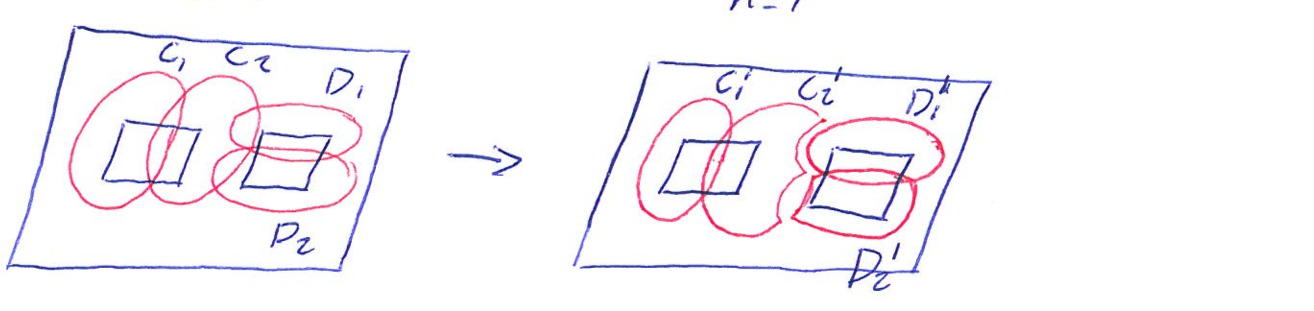
For every $x \in A$, $X - B$ is a nbh of x. By our prervious lemma, that implies there exists a nbh W of x s.t. $x \in W \subset \overline{W} \subset X - B$.
Since $\mathcal{B}$ is a basis and W_x is open, $\exists\, B_x \in \mathcal{B}$ s.t. B_x is a nbh of x andiscontained in W_x.
Hence $\{B_x\}_{x \in A}$ is a coven for A, but is countable so rename it $\{C_i\}_{i=1}^{\infty}$.
Repeat this process to construct a countable collection of basis elements ~~of~~ that cover B.
$\{D_i\}_{i=1}^{\infty}$.

For each n let

$$C_n' = C_n - \left(\bigcup_{i=1}^{n} \overline{D_i} \right) \quad \text{and} \quad D_n' = D_n - \left(\bigcup_{i=1}^{n} \overline{C_i} \right)$$

Let $C' = \bigcup_{n=1}^{\infty} C_n'$ and $D' = \bigcup_{n=1}^{\infty} D_n'$

Pic



Note: $A \subset C'$ and $B \subset D'$.

let $x \in A$, $x \in C_i$ for some i

If $x \in \bigcup_{j=1}^{i} \overline{D_i}$, then $x \in \overline{D_j}$ for some j.

However the D_j where chosen s.t. $\overline{D_j} \subset X - A$.

Thus $x \in C_i - \bigcup_{j=1}^{i} \overline{D_i}$, and $x \in C'$.

Similarly, $B \subset D'$.

~~Claim~~ $\cancel{B' \cap C' = \emptyset}$

~~If~~ $\cancel{x \in B' \cap C'}$, ~~then~~ $\cancel{x \in C_i' \cap X}$

Claim | $C' \cap D' = \emptyset$

If $x \in C' \cap D'$, then $x \in C_i' \cap D_j'$ for some i and j.

WLOG, assume $i \geq j$.

Hence, $x \in C_i' = C_i - \bigcup_{k=1}^{i} \overline{D_j}$ so, $x \notin D_j'$.

This is a contradiction.

So, $C' \cap D' = \emptyset$.

Hence, X is T_4. $\square$

A summary of results

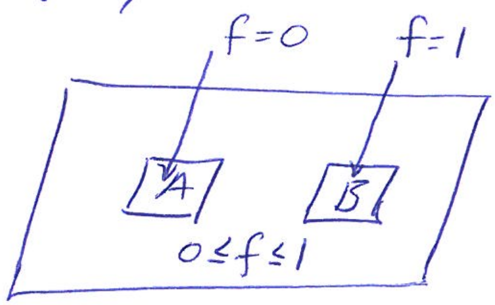
$$\begin{tikzcd}
\text{metrizable} \arrow[d, Rightarrow] & \\
T_4 \arrow[d, Rightarrow] & \\
T_3 \arrow[d, Rightarrow] \arrow[u, bend right, Rightarrow, "\text{and 2nd countable}"'] \arrow[uu, bend right=60, Rightarrow, "\text{Urysohn Metrization}"'] & \\
T_2 \arrow[d, Rightarrow] \arrow[uu, bend left, Rightarrow, "\text{and compact}"] & \\
T_1 &
\end{tikzcd}$$

Th^m (Urysohn Lemma)

Let X be T_4 and let A and B be disjiont closed subsets of X. There exists a continuous function $f: X \to [0,1]$ s.t.

$f(x)=0$ for every $x\in A$ and $f(x)=1$ for every $x\in B$.

Pic|



Sketch of a proof.|

Since X is T_4, ~~there exist disjiont open sets U_0 and U_1 s.t. $A\subset U_0$~~ and ~~$B\subset U_1$.~~ and $X-B=U_1$ is an open nbh of A, then there exist U_0 a nbh of A s.t.

$$A\subset U_0\subset \overline{U_0}\subset U_1$$

We want to define an open set U_p for each rational number s.t. if $p<q$ then $\overline{U_p} \subset U_q$.

We induct on the following list of rational numbers

$$\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \dots$$

Since U_1 is a nbh of the closed set $\overline{U_0}$, then there exis a nbh $U_{1/2}$ of ~~A~~ $\overline{U_0}$ s.t.

$$\overline{U_0} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_1.$$

Continue by induction to define U_p s.t. if $p<q$, then $\overline{U_p} \subset U_q$.

Next, define $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} \inf\{p \mid x \in U_p\} \\ 1 \quad \text{if } x \notin \bigcup_{p \in \mathbb{Q}} U_p \end{cases}$$

Note if $a \in A$ then $a \in U_0$, so $f(a) = 0$
if $b \in B$ then $b \in X - U_1$ so $f(b) = 1$.

<u>WTS f is continuous</u>

Let $f(x_0) \in (a,b) \subset \mathbb{R}$. WTS that there exists an ~~open set~~ nbh U_{x_0} s.t. $f(U_{x_0}) \subset (a,b)$.
Since $\mathbb{Q}$ is dence in $\mathbb{R}$, $\exists\, p, q \in \mathbb{Q}$ s.t.

$$a < p < f(x_0) < q < b$$

Turns out that $U_q - \overline{U_p}$ works. $\square$

Topology Day 25

Outline
- Urysohn Metrization Theorem

Recall|

Thm| If X is T_3 and second-countable, then X is T_4.

Thm| (Urysohn Lemma) If X is T_4 and A and B are disjiont ~~of~~ closed sets in X then there exists a continuous function $f: X \to [0,1]$ s.t.

$$f(a) = 0 \text{ for } a \in A \text{ and } f(b) = 1 \text{ for } b \in B.$$

Thm| (Urysohn Metrization)
If X is T_3 and second-countable, then X is metrizable.

Pf| We can assume X is T_4 by previous result.
Idea of proof: Show X is homeomorphic to a subspace of a metrizable space.

Step 1

Claim: There exists a countable collection of continuous functions $f_n : X \to [0,1]$ with the property that given any $x_0 \in X$ and any nbh U of x_0 there exists an index n s.t. $f_n(x_0) > 0$ and $f_n(y) = 0$ for all $y \in X - U$.

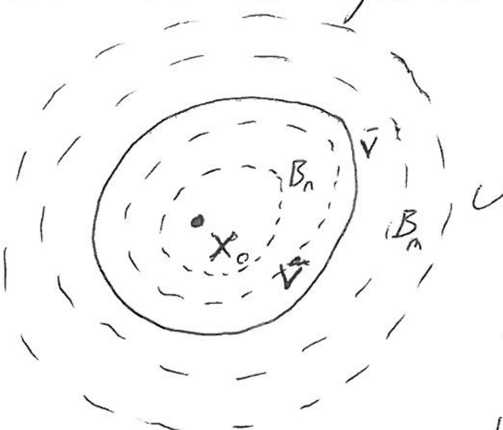
Note $\{x_0\}$ is closed and $X - U$ is closed, so by the Urysohn lemma there exists a continuous function $g : X \to [0,1]$ s.t. $g(x_0) = 1$ and $g(y) = 0$ for $y \in X - U$.

However, we need countably many g to distinguish all pairs (U, x_0). Must use second-countable!

Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for X.

For each pair n, m s.t. $\overline{B_n} \subset B_m$ we can apply Urysohn lemma ~~s.t.~~ to produce $g_{m,n} : X \to [0,1]$ with $g_{m,n}(x) = 0$ if $x \in \overline{B_n}$ and $g_{m,n}(x) = 1$ if $x \in X - B_m$.

This is enough to prove the claim.



- Pick B_m s.t. $x_0 \in B_m \subset U$
- By T_3, $\exists\, V$ a nbh of x_0 s.t. $\overline{V} \subset B_m$
- Pick B_n s.t. $x_0 \in B_n \subset V \subset \overline{V} \subset B_m \subset U$.
- Hence $g_{m,n}(x)$ has the desired property.
- Since $\{B_n\}_{n \in \mathbb{Z}^+}$ is countable $\{g_{m,n}\}_{m,n \in \mathbb{Z}^+ \times \mathbb{Z}^+}$ is countable.

Step 2 | Want to construct a homeomorphism from X into $[0,1]^\omega = \prod_{i=1}^{\infty} [0,1] \subset (\mathbb{R}^\omega, \text{uni})$ with the uniform metric topology.

Recall | the metric on $[0,1]^\omega$ is given by

$$d(\vec{x}, \vec{y}) = \sup\{|x_i - y_i|\}$$

Let $\{f_n\}_{n=1}^{\infty}$ be the collection constructed in step 1.

Define $F(x) = \left(f_1(x), \frac{f_2(x)}{2}, \frac{f_3(x)}{3}, \frac{f_4(x)}{4}, \dots\right)$

$F: X \to [0,1]^\omega$.

Claim | F is injective

By hausdorff.

$\exists\, n_0$ s.t. $f_{n_0}(x) = 1$ and $f_{n_0}(Y) = 0$

so $F(x) \neq F(Y)$.

Claim | F is open.

Pf | Tedious

Claim | F is continous.

Pf | Tedious (since we are working with uniform metric).

Thus, F is a homeomorphism onto its image.

So, X is metrizable. $\square$