MATH 550A - Spring 2014
Midterm One

Name: *Solutions*

PLEASE WRITE LEGIBLY AND EXPLAIN YOUR STEPS CAREFULLY, USING COMPLETE SENTENCES. No books and no notes. Remember to put your name at the top of this page. Good luck!

Problem	Score (out of)
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
Total	(50)

1.(10pts)

(a) Given topological spaces X and Y, define what it means for a function $f : X \to Y$ to be continuous.

Look in book.

(b) Give three additional conditions that are equivalent to the condition you provided in (a).

(c)Given a set X, define a basis for a topology on X.

2.(10pts) Let X be a topological space. Show X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)|x \in X\}$ is closed in $X \times X$.

$\Rightarrow$| Suppose X is hausdorfff. Let $(x,y) \in X \times X - \Delta$.
Since $(x,y) \notin \Delta$, then $x \neq y$. Since $x \neq y$ and X is hausdorff, then $\exists\, U_x$ a nbh of x and U_y a nbh of y s.t. $U_x \cap U_y = \emptyset$. Since U_x and U_y are open, then $U_x \times U_y$ is a nbh of (x,y) in $X \times X$.

Claim| $U_x \times U_y$ is disjiont from Δ.

Suppose not to form a contradiction. Let $(w,w) \in \Delta \cap U_x \times U_y$. then $w \in U_x$ and $w \in U_y$. This is a contradiction to $U_x \cap U_y = \emptyset$. So, $U_x \times U_y \subset X \times X - \Delta$.

Hence $X \times X - \Delta$ is open and Δ is closed. $\square$

$\Leftarrow$| Suppose Δ is closed. Then $X \times X - \Delta$ is open.
Let $x, y \in X$ s.t. $x \neq y$. Then $(x,y) \in X \times X - \Delta$.
By def of the product top. and def of open, there exists $U, V \subset X$ s.t. U and V are open and $(x,y) \in U \times V \subset X \times X - \Delta$. Clearly U is a nbh of x and V is a nbh of y.

Claim| $U \cap V = \emptyset$.

Suppose to form a contradiction that $w \in U \cap V$. Then $w \in U$ and $w \in V$. Hence $(w,w) \in U \times V$ and $\Delta \cap U \times V \neq \emptyset$. This contradicts $U \times V \subset X \times X - \Delta$.

Thus X is Hausdorff. $\square$

3.(10pts) Equip $\mathbb{R}^3$ with the dictionary order topology. Recall that in the dictionary ordering $(x, y, z) < (a, b, c)$ if $x < a$ or ($x = a$ and $y < b$) or ($x = a$ and $y = b$ and $z < c$).

(a) The subspace topology on the plane $z = 0$ is what familiar topology? Remember to justify your answer.

Under the dic order topology the set $\{(x_0, y_0, z) \mid -1 < z < 1\}$ is open for every fixed $x_0, y_0 \in \mathbb{R}$.
Let $(x_0, y_0, 0)$ be any point in the plain $z = 0$.
Since $\{(x_0, y_0, z) \mid -1 < z < 1\} \cap \{x, y, z \mid z = 0\} = (x_0, y_0, 0)$, then every point in the plane $z=0$ is open in the subspace topology. Hence, the topology is the discrete topology.

(b) The subspace topology on the plane $x = 0$ is what familiar topology? Remember to justify your answer.

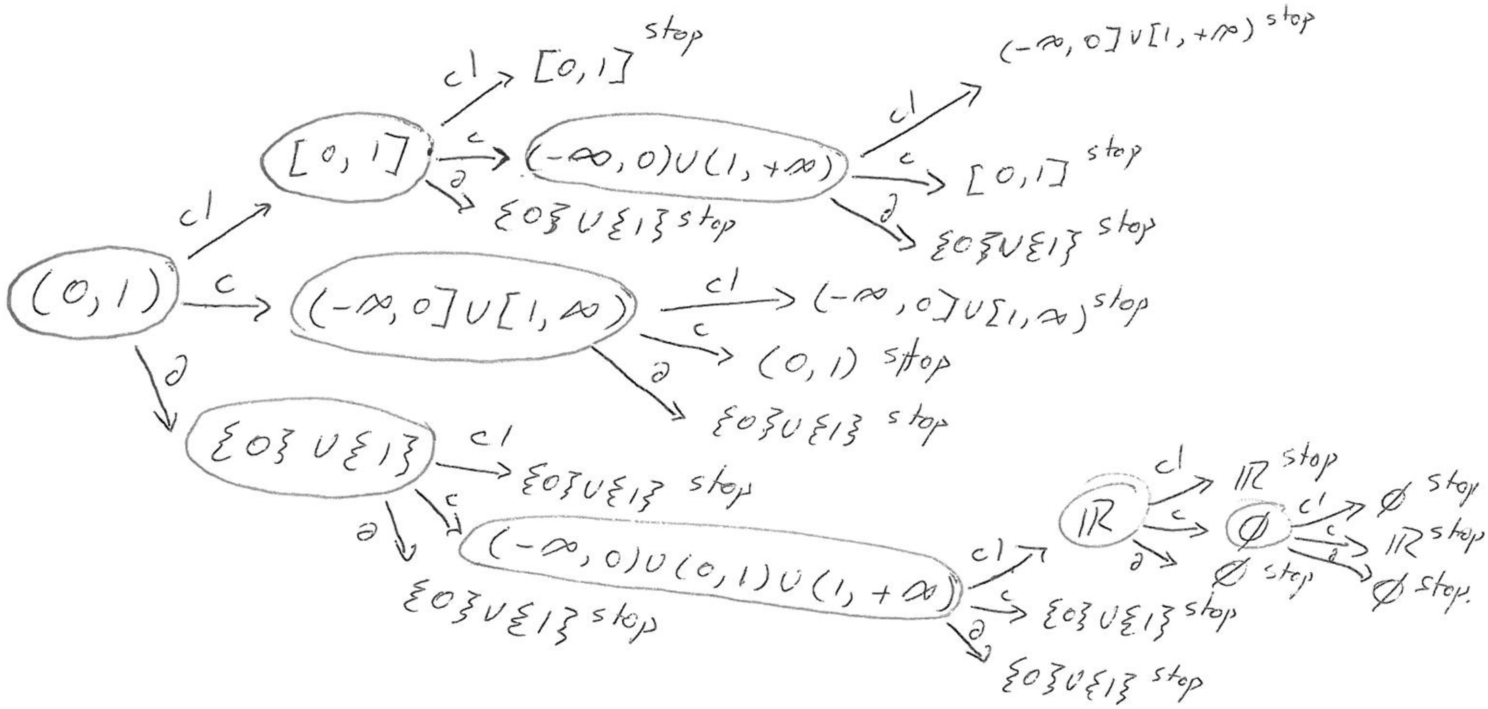
Basis elements for the dic. order topology on $\mathbb{R}^3$ meet the plone $x = 0$ in sets of the form $\{(0, y, z) \in \mathbb{R} \mid a < y < b\}$ or of the form $\{(0, y, z) \in \mathbb{R} \mid y = a \text{ and } b < z < c\}$
Together these sets form the basis of the dictionary order topology on the Plane $x = 0$.

4. (10pts) If $A \subset X$ where X is a topological space, we define the *boundary* of A by the equation

$$Bd(A) = \overline{A} \cap \overline{(X - A)}$$

Consider the collection of all subsets of a topological space X. The operations of closure $A \to \overline{A}$, complementation $A \to X - A$ and boundary $A \to Bd(A)$ are functions from this collection to itself. Find all sets that can be formed by applying any combination of these operations to $(0, 1) \subset \mathbb{R}$.

$A \xrightarrow{cl} \overline{A}$, $A \xrightarrow{c} X - A$, $A \xrightarrow{\partial} Bd(A)$



5. (10pts) Let $\mathbb{R}^2$ be the *set* of points in the plane and let $\mathbb{R}_s$ be the real numbers with the standard topology. Let $f : \mathbb{R}^2 \to \mathbb{R}_s$ such that $f((x, y)) = x$. What is the coarsest topology on $\mathbb{R}^2$ such that f is continuous? You should prove the collection of subsets you find is a topology and prove it is coarsest.

Claim: $\tau = \{ U \times \mathbb{R} \subset \mathbb{R}^2 \mid U \text{ is open in } \mathbb{R} \}$ is the coarsest topology s.t. f is continuous.

Subclaim τ is a topology

① $\emptyset \times \mathbb{R} = \emptyset$ and $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

② $\bigcup_{\alpha \in J} (U_\alpha \times \mathbb{R}) = (\bigcup_{\alpha \in J} U_\alpha) \times \mathbb{R} = V \times \mathbb{R}$ where V is open in $\mathbb{R}$.

③ $\bigcap_{i=1}^{n} (U_i \times \mathbb{R}) = (\bigcap_{i=1}^{n} U_i) \times \mathbb{R} = V \times \mathbb{R}$ where V is open in $\mathbb{R}$.

Hence τ is a topology.

Subclaim $f : (\mathbb{R}^2, \tau) \to \mathbb{R}$ is continuous.

Let $U \subset \mathbb{R}$ be open $f^{-1}(U) = U \times \mathbb{R} \in \tau$. Hence f is continuous

Subclaim If $f : (\mathbb{R}^2, \tau') \to \mathbb{R}$ is continuous, then $\tau \subset \tau'$

Let $U \times \mathbb{R} \in \tau$. Then $U \subset \mathbb{R}$ is open. Since f is continuous, $f^{-1}(U) = U \times \mathbb{R}$ is an element of τ'. Thus $\tau \subset \tau'$. $\square$