Homework 6

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#1a. Show that if A is a deformation retract of X, then A is homotopic to X.

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Pf - Let $A \subseteq X$, and let A be a deformation retract of X.

Then $\exists$ a continuous map $H: X \times I \to A$ s.t. $H(x,0) = x$

$H(x,1) \in A$

$H(a,t) = a \quad \forall a \in A$

Now, consider the continuous retraction map $r: X \to A$ defined by $r(x) = H(x,1)$ s.t. $r|_A = id_A$

and the inclusion map $i: A \to X$ st $i(a) = a$.

(WTS: $i \circ r \simeq id_X$ and $r \circ i \simeq id_A$)

$i \circ r$ and $r \circ i$ are continuous because they are the composition of cont. func.

$r \circ i: A \to A$ via $r(i(a)) = r(a) = id_A$

Now, $id_X \simeq i \circ r$ by $H: X \times I \to A$ from above

H is continuous

$H(x,0) = x = id_X$

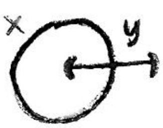
$H(x,1) = r(x) = i(r(x))$.

Hence A is homotopic to X.

#1b. Let $X = S^1$ and $Y = \{(x,y) \mid x^2+y^2=1\} \cup \{(x,y) \mid y=0, 1<x<2\}$.

Show that X is homotopic to Y.

Pf - By part a, if X is a deformation retract of Y, then X is homotopic to Y.



So, want to show X is a deformation retract of Y.

Consider $H: Y \times I \to Y$ (where $Y \subset \mathbb{R}^2$) defined by

$$H((a,b),t) = \begin{cases} (a,b) & (a,b) \in X \\ (a,b)(1-t) + \dfrac{t(a,b)}{\sqrt{a^2+b^2}} & (a,b) \in (0,2) \times \{0\} \end{cases}$$

H is continuous by the pasting lemma since $(a,b) = id_X$ is continuous

$(a,b)(1-t) + \dfrac{t(a,b)}{\sqrt{a^2+b^2}}$ is continuous ($(0,0)$ not in domain & scalar mult. & add is continuous),

and for $(a,b) \in X \cap (0,2) \times \{0\}$ is $(1,0)$, $H((1,0),t) = \begin{cases} (1,0) \\ (1,0)(1-t) + t(1,0) = (1,0) \end{cases}$ agree!

Also, $H((a,b),0) = (a,b) \quad \forall (a,b) \in Y$

$$H((a,b),1) = \begin{cases} (a,b) & (a,b) \in X \\ \dfrac{(a,b)}{\sqrt{a^2+b^2}} & (a,b) \in (0,2) \times \{0\} \end{cases} \in X$$

AND, let $(a,b) \in X$, $H((a,b),t) = (a,b)$

$\therefore$ X is a deformation retract of Y & $X \simeq Y$ b

#2 Use van Kampen's theorem and induction to show that $\pi_1(S^n, x_0) \cong \{1\}$ for $n \geq 2$.

Hint: S^n can be constructed by gluing two n-balls together along their boundary.

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pf - Let $\varepsilon > 0$

For $n=2$, define S^2 as $S^2 = A \cup B$ where

$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \text{ and } z > -\varepsilon\}$ and

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \text{ and } z < \varepsilon\}$

Note that A and B are each path-connected and open in $\mathbb{R}^3$.

Also note that $A \cap B \cong S^1 \times (-\varepsilon, \varepsilon)$, which is also path-connected & open.

So, by van Kampens theorem $\pi_1(S^2, x_0) \cong \dfrac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\ker(\varphi)}$

$A \cong D^2$ (Flattening the upper hemisphere of a basketball into a solid circle)

$B \cong D^2$

And $\pi_1(D^2, x_0) = \{1\}$ since D^2 is convex

thus, $\pi_1(S^2, x_0) = \dfrac{\{1\} * \{1\}}{\ker(\varphi)} = \{1\}$ regardless of $\ker(\varphi)$.

Now, assume $\pi_1(S^n, x_0) = \{1\}$ for some $n > 2$

Next, consider $n+1$.

Define S^{n+1} as $S^{n+1} = A \cup B$ where

$A = \{(x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \mid \sum_{i=1}^{n+2} x_i^2 = 1 \text{ and } x_{n+2} > -\varepsilon\}$ and

$B = \{(x_1, x_2, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \mid \sum_{i=1}^{n+2} x_i^2 = 1 \text{ and } x_{n+2} < \varepsilon\}$.

Note that A and B are each path connected and open in $\mathbb{R}^{n+2}$.

Also note that $A \cap B = S^n \times (-\varepsilon, \varepsilon)$, which is also path connected & open.

So, by van Kampen's theorem, $\pi_1(S^{n+1}, x_0) = \dfrac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\ker(\varphi)}$

$A \cong B \cong D^{n+1}$

And $\pi_1(D^{n+1}, x_0) = \{1\}$ since D^{n+1} is convex.

Thus, $\pi_1(S^{n+1}, x_0) = \dfrac{\{1\} * \{1\}}{\ker(\varphi)} = \{1\}$ regardless of $\ker(\varphi)$.

Therefore, $\forall n \geq 2$, $\pi_1(S^n, x_0) \cong \{1\}$

[illegible]

$\frac{7}{10}$

#3. Hatcher pg 53 ex 8 : Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Pf – Consider X = two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other

$X \cap (\mathbb{R}^2 \times (-\varepsilon, \infty))$

Define $A = \{X \cap (-\varepsilon, \infty)\} \cong$ $\times S^1$

$B = \{X \cap (-\infty, \varepsilon)\} \cong$ $\times S^1$

Notice that A, B are path connected and open in $\mathbb{R}^3$,

$A \cup B = X$ and X is open since it is the union of 2 open sets

$A \cap B = X \cap (-\varepsilon, \varepsilon)$ is path connected and open since the intersection of two open spaces is open. $A \cap B \simeq S^1$

Thus, by van Kampen theorem $\pi_1(X, x_0) \cong \dfrac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\ker(\varphi)}$

Notice that $\exists$ a deformation retract of A onto $S^1 \times S^1$, so by #1, $A \simeq S^1 \times S^1$

Further, by theorem 58.7, $\pi_1(A) \cong \pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$

Similarly, $\pi_1(B) \cong \mathbb{Z} \oplus \mathbb{Z}$

Now, $\exists$ a deformation retract of $A \cap B$ onto $S^1 \times \{x_0\}$

so, by theorem 58.7 $\pi_1(A \cap B) \cong \pi_1(S^1 \times \{x_0\}) = \mathbb{Z}$

$\pi_1(A)$ has group presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$

$\pi_1(B)$ has group presentation $\langle c, d \mid cdc^{-1}b^{-1} \rangle$

$\pi_1(A \cap B)$ has group presentation $\langle w \mid \ \rangle$

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dr, hook, "i_A"] & \\
\pi_1(A \cap B) \arrow[ur, "j_A"] \arrow[dr, "j_B"'] & & \pi_1(X) \\
 & \pi_1(B) \arrow[ur, hook, "i_B"'] &
\end{tikzcd}$$

You should talk about what choices you made so that this is true.

$j_A(w) = j_A(S^1 \times \{x_0\}) = a$; $j_B(w) = j_B(S^1 \times \{x_0\}) = c \Rightarrow (j_B(w))^{-1} = c^{-1}$

Thus, by van Kampen, $\pi_1(X, x_0) = \dfrac{\langle a, b \mid aba^{-1}b^{-1} \rangle * \langle c, d \mid cdc^{-1}d^{-1} \rangle}{\langle\langle a c^{-1} \rangle\rangle}$

$= \langle a, b, c, d \mid aba^{-1}b^{-1}, cdc^{-1}d^{-1}, ac^{-1} \rangle$

$= \langle a, b, d \mid aba^{-1}b^{-1}, ada^{-1}d^{-1} \rangle$

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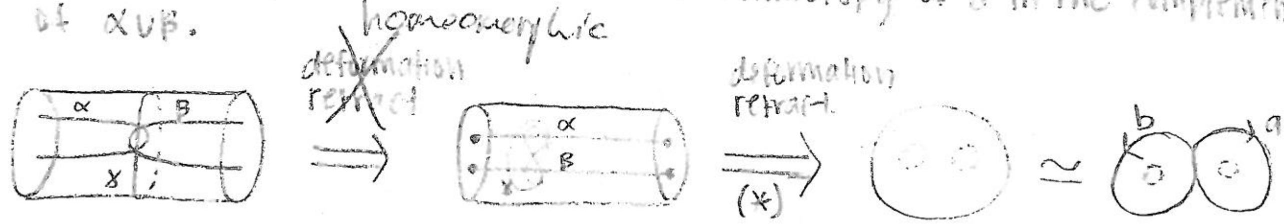
#4. Hatcher pg 53 ex 10: consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.



(*) Let X be a topological space.
Then $X \times \{0\}$ is a deformation retract of $X \times I$

Pf - consider $H: (X \times I) \times I \to X \times I$ defined by $H((x,t),s) = (x, (1-s)t$

H is continuous because $(x,t) \to x$ is a projection map, which is continuous, $(1-s)t$ is a continuous polynomial & the product of continuous functions is continuous, thus H is continuous

$H((x,t),0) = (x,t)$

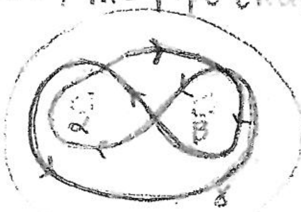
$H((x,t),1) = (x,0) \in X \times \{0\}$

$H((x,0),s) = (x,0)$

Thus H is a deformation retraction of $X \times I$ onto $X \times \{0\}$

$\pi_1(\infty) \cong F_2$ (Free group of 2 degrees) $= \langle ab \mid \ \rangle$

Using the pipe cleaners to keep track of γ during the deformations, we see



Thus, compared to (b, a figure-eight), $\gamma = aba^{-1}b^{-1}$

However, every word is non-trivial in F_2

~~But γ is non-trivial, so we have a contradiction.~~

~~Thus, γ is not null homotopic.~~

use a theorem from the book