Announcements

- HW due on Tuesday
- Approve a topic with me by Thursday

Outline

- Calculations using van Kampen's theorem

Recall

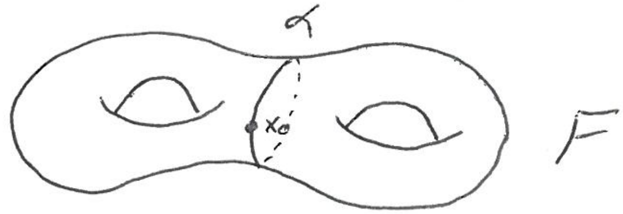
Let X be the union of two open, path-connected subsets A and B s.t. $A \cap B$ is path-connected.

$$\begin{tikzcd} & A \arrow[dr, hook] & \\ A \cap B \arrow[ur, hook] \arrow[dr, hook] & & A \cup B = X \\ & B \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(A) \arrow[dr, "i_A"] & \\ \pi_1(A \cap B) \arrow[ur, "j_A"] \arrow[dr, "j_B"'] & & \pi_1(X) \\ & \pi_1(B) \arrow[ur, "i_B"'] & \end{tikzcd}$$

There exists a unique extension of i_A and i_B to a map $\varphi : \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(X)$.

By van Kampen's theorem φ is onto and $\operatorname{Ker} \varphi$ is generated by elements of the form $j_A(\omega)(j_B(\omega))^{-1}$ where $\omega \in \pi_1(A \cap B)$.

Ex | Show the loop α in the genus 2 surface F is not ~~to~~ null homotopic.



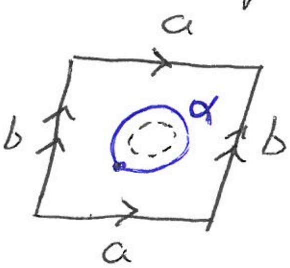
Step 1: Express α as a product of generators of $\pi_1(F)$.

Step 2: Show the product of generators is non-trivial in $\pi_1(F)$.

Step 1 |

Recall that we decomposed F as $A \cup B$

Where $A \cong$



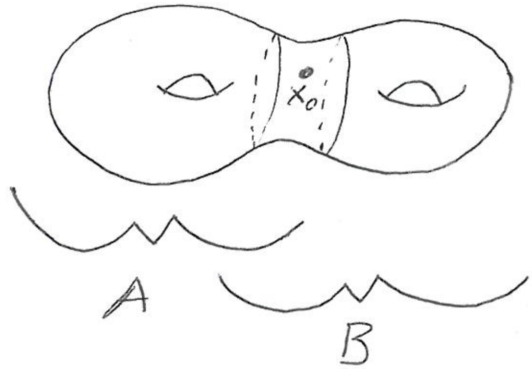
Additionally $\pi_1(F) \cong \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

So, $\alpha = aba^{-1}b^{-1} \in \pi_1(F)$.

Step 2 | Since $aba^{-1}b^{-1} \neq (aba^{-1}b^{-1}cdc^{-1}d^{-1})^n$ for any $n \in \mathbb{Z}$, then $aba^{-1}b^{-1}$ is not the identity element of $\pi_1(F)$

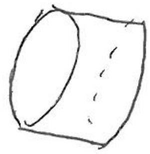
Hence α is not ~~to~~ null homotopic.

Ex| Let X be the genus 2 surface



$A =$  $\cong S^1 \times S^1 - D^2$

$B =$  $\cong S^2 \times S^1 - D^2$

$A \cap B =$  $\cong S^1 \times (0,1)$

Since A, B and $A \cap B$ are all open, path-connected subsets of X, we can apply Van Kampens theorem.

$$\pi_1(X, x_0) \cong \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\operatorname{Ker}(\varphi)}$$

Must find these.

Note: A and B deformation retract onto the wedge of two circles (ie $S^1 \vee S^1$ ∞)

Note: $S^1 \vee S^1$ is homotopic to the eye glass graph

Proof: Exercise

Hence $\pi_1(A, x_0) \cong \pi_1(B, x_0) \cong \pi_1(\infty) \cong \pi_1(\text{O-O}) \cong F_2$

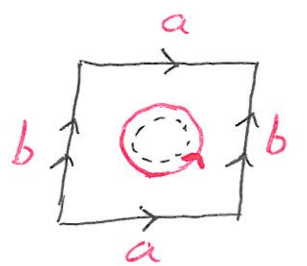
Since $S^1 \times (0,1)$ deformation retracts onto $S^1 \times \{1/2\}$, then $\pi_1(S^1 \times (0,1)) \cong \pi_1(A \cap B, x_0) \cong \mathbb{Z} \cong \langle w \rangle$

Recall the ~~induced~~ maps induced by the inclusion maps

$$j_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0) \quad \text{and} \quad j_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$$

$$j_A : \langle w \rangle \to \langle a, b \mid \rangle \quad \text{and} \quad j_B : \langle w \rangle \to \langle c, d \mid \rangle$$

We need to figue out what these maps do!



$$j_A(w) = aba^{-1}b^{-1}$$

$$(j_B(w))^{-1} = cdc^{-1}d^{-1}$$

$$\text{So, } \pi_1(X, x_0) \cong \frac{\langle a, b \mid \rangle * \langle c, d \mid \rangle}{\langle\langle aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle\rangle}$$

$$\cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

Problem (really a group theory problem)

Show the genus 2 surface is not homotopic to the torus $S^1 \times S^1$. (Hint: If two groups are isomorphic, then their abelianizations are isomorphic).

Ex | Show the genus 2 surface is not homotopic to the torus.

Let F be the genus 2 ~~sh~~ surface.

Let T be the torus.

We know $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ and

$$\pi_1(F) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

How do we show these groups are different?

Abelianize!!

The abelianization of $\mathbb{Z} \oplus \mathbb{Z}$ is $\mathbb{Z} \oplus \mathbb{Z}$.

The abelianization of $\pi_1(F)$ is

$$\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \cup aba^{-1}b^{-1}, aca^{-1}c^{-1}, ada^{-1}d^{-1}, bcb^{-1}c^{-1}, bdb^{-1}d^{-1}, cdc^{-1}d^{-1} \rangle$$

$\cong$ the free abelian group on 4 generators

$\cong \bigoplus_{i=1}^{4} \mathbb{Z}$

By the fundumental theorem of finitely generated abelian groups $\bigoplus_{i=1}^{2} \mathbb{Z} \not\cong \bigoplus_{i=1}^{4} \mathbb{Z}$.

So, $\pi_1(T) \not\cong \pi_1(F)$.

Announcements

- HW 6 due on Thursday
- Project topic due on Thursday
  - New topics
    - Proof of Von Kampen's theorem and applications.

Outline

- Introduction to homology

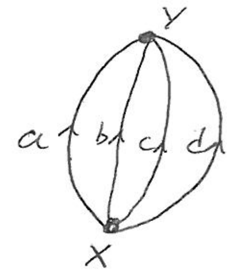
Limitations of the fundamental group.

- From HW, $\pi_1(S^n) \cong \pi_1(S^m)$ if $n, m \geq 2$.
- If X is an n-manifold with $n \geq 3$, then
  $\pi_1(X - pt) \cong \pi_1(X)$.

Because the fundamental group is defined using loops, it cannot "see" higher dimensional topological structure.

However, "homology groups", denoted $H_n(X)$, will be able to see much of this higher-dimensional structure.

Initial, motivating example

Examine the following graph  $= G$

You can show $\pi_1(G) \cong F_3$ and generated by loops $a * \bar{b}$, $b * \bar{c}$ and $c * \bar{d}$.

Lets think of loops as a linear combination of edges a, b, c, d. So, we are "abelianizing" $\pi_1(X)$.

$$a * \bar{b} \to a - b$$
$$b * \bar{c} \to b - c.$$

when a linear combination of edges represents a loop we call it a <u>cycle</u>. How do we find cycles?

<u>Ex</u> $2a - b$ is <u>not</u> a cycle

$2a - 2b + c + d$ is a cycle $\quad a * \bar{b} * a * \bar{b} * c * \bar{d}$

<u>Ex</u> Cycles no longer have the problem of base point

$$-b + a = a - b \qquad a * \bar{b} \neq \bar{b} * a.$$

<u>Ex</u> $ka + lb + mc + nd$ is a cycle if

① The number of times it exists y = # of times enters.

② Same for the vertex X.

Because of the orientations ① $\Rightarrow k + l + m + n = 0$ and

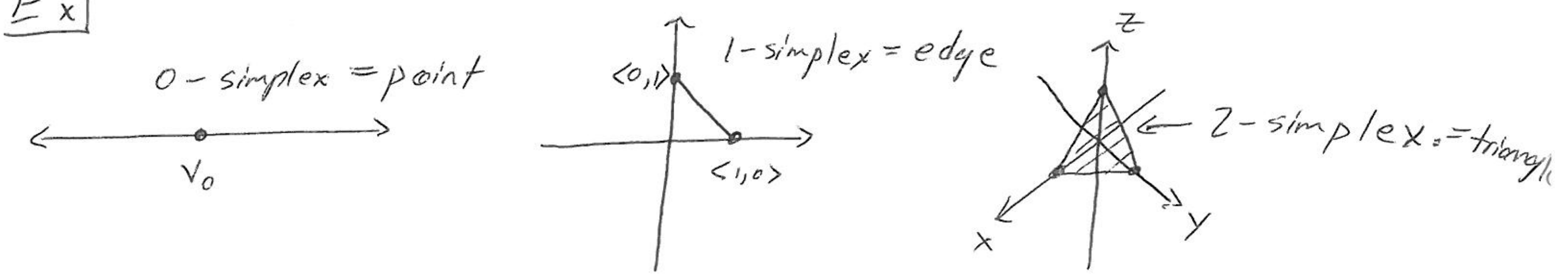
② $\Rightarrow -k - l - m - n = 0$

This is equivalent to the following more algebraic description.

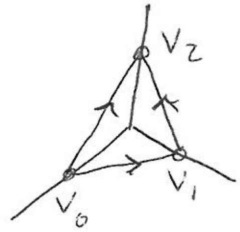
More precisely:

Def) an *n-simplex* is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, v_1, \ldots, v_n$ s.t. these points do not lie in a hyperplane of dimension less than n.

Ex)



We denote the simplex by the ordered set of vertices $[v_0, v_1, \ldots, v_n]$. Note, by ordering the vertices we induce an ordering on the edges. i.e.



$[v_0, v_1, v_2]$

Def) A face of $[v_0, \ldots, v_n]$ is and sub simplex with vertices a sub set of $\{v_0, \ldots, v_n\}$ and the vertices of a face are always ordered according to the original ordering.

Let $C_1 = \underset{\langle a \rangle}{\mathbb{Z}} \oplus \underset{\langle b \rangle}{\mathbb{Z}} \oplus \underset{\langle c \rangle}{\mathbb{Z}} \oplus \underset{\langle d \rangle}{\mathbb{Z}}$

$C_0 = \underset{\langle x \rangle}{\mathbb{Z}} \oplus \underset{\langle y \rangle}{\mathbb{Z}}$

Let $\partial : C_1 \to C_0$ s.t. $\partial = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

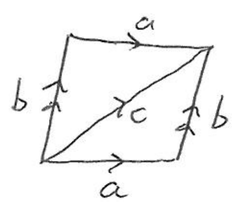
$X \in C_1$ is a cycle if $\cancel{\partial(x)}$ $x \in \ker(\partial) \cong \mathbb{Z}^3$

It will turn out that $H_1(X) \cong \mathbb{Z}^3$.

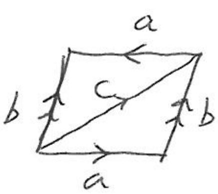
Δ-complexes!

As we have seen, many surfaces (in fact all of them) can be constructed by gluing triangles together along their edges.

i.e.

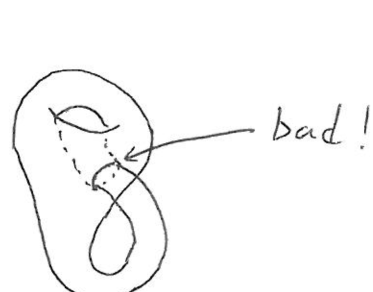


$= S^1 \times S^1 = \text{torus}$



$= S^1 \tilde{\times} S^1 = \text{klein bottle}$

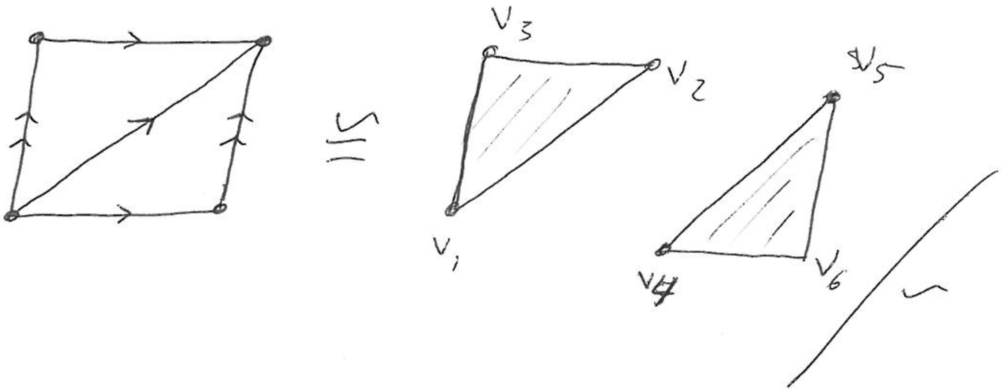
Top. spaces that can be constructed by taking edges, triangle, tetrahedra and their higher dimensional analongs and gluing them together along their faces will be called Δ-complexes.



klein bottle.

Def A <u>Δ-complex</u> is the quotient space of a collection of disjiont simplices obtained by identifiying certain faces via canonical linear homomorphisms that <u>preserve the ordering of the vertices</u>.

Ex



Def ~~A simple~~ An n-simplex with all of its proper faces deleted is called an <u>open n-simplex</u>. ~~Given a Δ-complex X containing an n-simple, ζ_α^n, the open n-simplex e_α^n corresponding to ζ_α^n~~

Def Given a Δ-complex X, let $\Delta_n(X)$ be the free abelian group with basis the open n-simpleces e_α^n of X. Elements of $\Delta_n(X)$ are called n-chains. and can be written $\sum_\alpha n_\alpha e_\alpha^n$ where $n_\alpha \in \mathbb{Z}$.

The boundary map

Given a n-simplex Δ^n, the boundary of Δ^n will be a signed linear combination of the $(n-1)$-dimensional faces of Δ^n

$$\partial([v_0, v_1, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_1, \dots, \hat{v_i}, \dots, v_n]$$

this notation means "remove this vertex"!

Ex

$$\partial([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

This boundary map can be extended in a natural way to a map $\partial_n : \Delta_n(X) \longrightarrow \cancel{\partial_n}\Delta_{n-1}(X)$.