Outline

- A quick review from last time
- Paths which are not path-homotopic
- Products of paths
- The algebra of the product of path-homotopy classes

Review

Given $f: X \to Y$ and $g: X \to Y$ continuous maps, $f \simeq g$ if there exists a continuous map $H: X \times I \to Y$ s.t. $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Note: We can think of H as a "movie" taking f to g.

Given paths $f: I \to Y$ and $g: I \to Y$, $f \simeq_p g$ if there is a homotopy $H: \cancel{X} \times I \to Y$ s.t.

$$H(0,t) = x_0 \text{ for all } t \text{ and}$$

$$H(1,t) = x_1 \text{ for all } t.$$

A very useful prop.

<u>Prop</u> Let $A \subset \mathbb{R}^n$ be a convex subset and let $f: X \to A$ and $g: X \to A$ be continuous maps, then $f \simeq g$. (Also true for path-homotopy if f and g both begin and end at the same points)

<u>Cor</u> Any two paths in $\mathbb{R}^n$ are homotopic.

Show homotopy of paths **video**

<u>Claim</u> Any two pathes in $\mathbb{R}^2$ are homotopic.

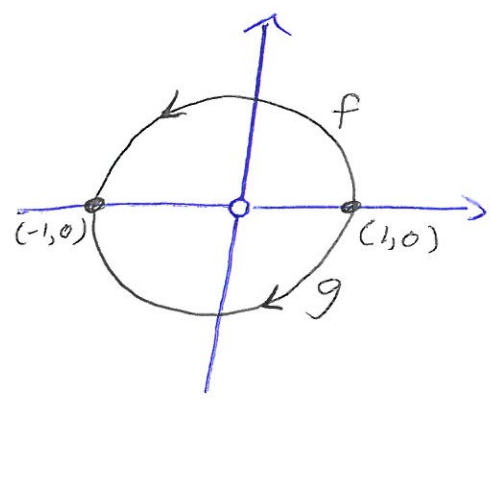
(not true if we replace homotopic with path homotopic)

<u>Pf</u> Let $f: I \to \mathbb{R}^2$ and $g: I \to \mathbb{R}^2$ be pathes.
Let $F: I \times I \to \mathbb{R}^2$ s.t.

$$F(t,s) = (1-s)f(t) + s\,g(t).$$

Show F is a homotopy between f and g.
You may assume that scalar multiplication and vector addition are continuous functions.

<u>Example</u> The following two pathes are <u>not</u> path homotopic in $\mathbb{R}^2 - \{0\}$. (Hard to prove, but intuitively true)



$f(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \pi$

$g(t) = \langle \cos(t), -\sin(t) \rangle$ for $0 \le t \le \pi$

Product of Paths

Let X be a top space and $x_0, x_1, x_2 \in X$. Let f be a path in X from x_0 to x_1 and let g be a path in X from x_1 to x_2. Define

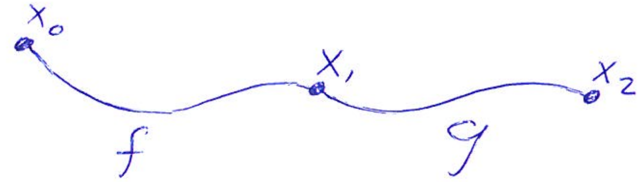
$$f * g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

Note: $f * g(s)$ is well defined since

$$f(2(1/2)) = f(1) = x_{\phi}$$

and $g(2(1/2)-1) = g(0) = x_1$.

Also, $f * g(s)$ is continuous by the pasting lemma.



Hence, $f * g(s)$ is a path in X from x_0 to x_2.

Let $[f]$ and $[g]$ be path homotopy classes in X. Whenever $f * g$ is defined, define

$$[f] * [g] = [f * g].$$

Prop] The operation $[f]*[g]$ is well defined.

Proof] Let f and f' be paths in X from x_0 to x_1 s.t. $f \simeq_p f'$.
Let g and g' be paths in X from x_1 to x_2 s.t. $g \simeq_p g'$.
WTS $f*g \simeq_p f'*g'$.

Let $F: I\times I \to X$ be the path homotopy from f to f'.
Let $G: I\times I \to X$ be the path homotopy from g to g'.

Define $H: I\times I \to X$ by

$$H(s,t) = \begin{cases} F(2s,t) & 0 \le s \le 1/2 \\ G(2s-1,t) & 1/2 \le s \le 1 \end{cases}$$

Clam: H is a path homotopy between $f*g$ and $f'*g'$.

1. H is continuous by pasting lemma.
2. $H(0,t) = F(0,t) = x_0$
3. $H(1,t) = G(1,t) = x_2$
4. H is well defined since $F(2(1/2),t) = x_1 = G(2(1/2)-1,t)$
5. $H(s,0) = F(2s,0) * G(2s-1,0) = f*g(s)$
6. $H(S,1) = f'*g'(s)$.

Thm | (51.2) The opperation $*$ on path-homotopy classes of paths is

① Associative

② has left and right ~~inverses~~ identities

③ has inverses.

Proof |

First we prove ②.

If $x \in X$, let e_x denote the constant path s.t.

$e_x(t) = X$. for all $t \in I$.

WTS that if f is a path in X from x_0 to x_1, then

$$[e_{x_0}] * [f] = [f] \text{ and } [f] * [e_{x_1}] = [f].$$

First, we defi~~ne~~ useful paths in I.

$e_0 : I \to I$ s.t. $e_0(s) = 0$ for all $s \in I$.

$i : I \to I$ s.t. $i(s) = s$ for all $s \in I$

Since I is convex, there is a straight-line path homotopy from i to $e_0 * i$. given by $G : I \times I \to I$.

Let $f : I \to X$ be any path from x_0 to X_1

Since f and G are continuous $f \circ G : I \times I \to I$ is a path homotopy from $f(i(s))$ to $f(e_0 * i(s))$

However $f(i(s)) = f(s)$ and $f(e_0 * i(s)) = e_{x_0} * f(s)$

Hence $[e_{x_0}] * [f] = [f]$. By a similar argument $[f] * [e_{x_1}] = [f]$

③ Given a path $f: I \to X$ from x_0 to x_1, let $\bar{f}: I \to X$ be the path defined by $\bar{f}(t) = f(1-t)$.

WTS $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$

Note that $i * \bar{i}$ is a path in I from 0 to 0. Since I is convex, there is a path-homotopy G between $i * \bar{i}$ and e_0.

Let f be a path in X from x_0 to x_1. Note that $f \circ G$ is a path-homotopy in X between $f(i * \bar{i})$ and $f(e_0)$. But,

$$f(i * \bar{i}(t)) = f * \bar{f}(t) \quad \text{and} \quad f(e_0) = e_{x_0}$$

Hence, $[f] * [\bar{f}] = [e_{x_0}]$

A similar argument shows $[\bar{f}] * [f] = [e_{x_1}]$.

① (Associativity is the most dificult).

Given pathes f, g, h in X.

$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

whenever the left and right handsides are well defined expresions.

First we define a useful map.

Given two intervals $[a,b]$ and $[c,d]$ in $\mathbb{R}$.

Out of Hatcher page 27.

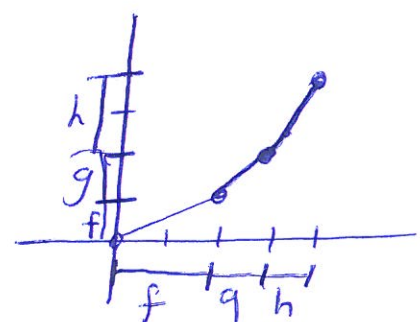
Let $f: I \to X$ be a path and $\varphi: I \to I$ be a continuous function s.t. $\varphi(0) = 0$ and $\varphi(1) = 1$.

Claim: $f \simeq_p f \circ \varphi$ via the path homotopy $f \circ \varphi_s$ where

$$\varphi_s(t) = (1-s)\varphi(t) + ts$$

Note: This claim follows imediately from our proposition regarding straight line homotopies on convex sets.

Define $\varphi(t) = \begin{cases} \cancel{\tfrac{1}{4}}\,\tfrac{1}{2}t & 0 \le t \le 1/2 \\ t - 1/4 & 1/2 \le t \le 3/4 \\ 2t-1 & 3/4 \le t \le 1 \end{cases}$



Given ~~Fo~~ paths $f, g, h: I \to X$ s.t. $f(1) = g(0)$ and $g(1) = h(0)$ we want to show

$$f * (g * h) \simeq_p (f * g) * h$$

By our previous observation $\cancel{f*(g*h)(\varphi(t)) \simeq_p f*(g*h)(t)}$ $(f*g)*h(\varphi(t)) \simeq_p (f*g)*h(t)$

However, $\cancel{f*(g*h)(\varphi(t)) = (f*g)*h(t).}$ $(f*g)*h(\varphi(t)) = f*(g*h)(t)$

Hence $[f]*([g]*[h]) = ([f]*[g])*[h]$. $\square$

The fundamental Group

Recall: A <u>group</u> is a set G together with a binary operation $\circ$ s.t. the following hold.

1. $\forall a, b \in G$, $a \circ b \in G$
2. $\forall a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.
3. $\exists e \in G$ s.t. for every $a \in G$ $e \circ a = a \circ e = a$.
4. For every $a \in G$ there exists $b \in G$ s.t. $a \circ b = b \circ a = e$.

<u>Def</u> Let X be a top space and $x_0 \in X$. A path in X that begins and ends at x_0 is called a <u>loop</u> in X based at x_0. The set of <u>path-homotopy classes</u> of loops in X based at x_0 under the operation $*$ is called the <u>fundamental group</u> of X relative to the <u>base point</u> x_0. It is denoted $\pi_1(X, x_0)$

<u>Example</u> Find $\pi_1(\mathbb{R}^2, (0,0))$

Let f and g be any two loops in $\mathbb{R}^2$ based at $(0,0)$. by our privious prop. $f \simeq_p g$.
Hence $|\pi_1(\mathbb{R}^2, (0,0))| = 1$ and
$\pi_1(\mathbb{R}^2, (0,0)) \cong \mathbb{I}$ (the trivial group). $\square$