Announcements

- New HW
- Colloquium

Outline

- Review of last time
- Review of group theory
- The fundamental group.

Review of last time

Let X be a top space and let $[f]$ and $[g]$ be path homotopy classes of paths f and g respectively.

Define $[f]*[g] = [f*g]$

Thm (51.2) The operation $*$ on path-homotopy classes has the following properties

1) $*$ is associative.
2) has left and right identities.
3) has inverses.

Fill in the blank given f a path from x_0 to x_1

$$[f] * \_\_\_ = [f]$$

$$\_\_\_ * [f] = [f]$$

$$[f] * \_\_\_ = \text{left identity}$$

$$\_\_\_ * [f] = \text{right identity}$$

Review of group theory

A group is a set G together with the binary operation $*$ s.t. the following group axioms hold.

① closure : For all $a, b \in G$, $a * b \in G$.

② associativity : For all $a, b, c \in G$

$$a * (b * c) = (a * b) * c$$

③ Identity : $\exists\, e \in G$ s.t. $\forall a \in G$ $a * e = e * a = a$.

④ Inverse : $\forall a \in G$ $\exists\, \bar{a} \in G$ s.t. $a * \bar{a} = \bar{a} * a = e$.

Def | Let G and G' be groups. A map $f: G \to G'$ is a homomorphism if for all $x, y \in G$

$f(x * y) = f(x) * f(y)$.

Ex | $f: (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$

$$f(x) = e^x$$

$$f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \cdot f(y).$$

Def | Given a homomorphism $f: G \to G'$ the kernel of f (denoted $\ker(f)$) is a subgroup of G given by

$$\ker(f) = \{x \in G \mid f(x) = e\}$$

Def | A bijective homomorphism is an isomorphism.

Exercise | Show $f: (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ given by $f(x) = e^x$ is an isomorphism.

Let $H < G$ be a subgroup. A left coset of H in G is a set $xH = \{xy \mid y \in H\}$.

Define right coset similarly.

A subgroup $H < G$ is <u>normal</u> if
$\{xyx^{-1} \mid y \in H\} = H$ for all $x \in G$.

Recall the <u>quotient of G by H</u> (denoted G/H) is the set of left cosets of H under the opperation $(xH)\cdot(yH) = (x\cdot y)H$.

Note: If H is a normal subgroup of G then G/H is a group.

<u>The fundamental Group</u>

<u>Def</u> Let X be a top. space and let $x_0 \in X$. A path in X that begins and ends at x_0 is called a <u>loop in X based at x_0</u>. The set of path homotopy classes of loops based at x_0 under the opperation $*$ is called the fundamental group of X based at x_0 and is denoted $\pi_1(X, x_0)$.

Note: The fact that this is a group follows imediatedy from Thm 51.1 and the following observations. Since loops begin and end at a common point, left and right inverses are equal. By restricting to loops ~~based~~ based at x_0 the operation $*$ is always defined.

<u>Ex</u> Let $A \subset \mathbb{R}^n$ be a convex subset of $\mathbb{R}^n$. Let $a_0 \in A$. Show $\pi_1(A, a_0) \cong \mathbb{1}$.

Recall the following prop.

<u>Prop</u> Let $A \subset \mathbb{R}^n$ be convex and ~~the~~ let $f: I \to A$ and $g: I \to A$ be paths s.t. $f(0) = g(0)$ and $f(1) = g(1)$, then $f \simeq_p g$.

Let $f: I \to A$ be any loop ~~from a~~ based at a_0. By the prop $f \simeq_p e_{a_0}$ where e_{a_0} is the constant path at a_0.

By transitivity of path homotopy

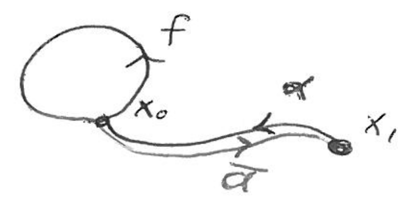
$$\pi_1(A, a_0) \cong \{[e_{a_0}]\} \cong \mathbb{1}. \quad \square$$

<u>Question</u> How much does the x_0 in $\pi_1(X, x_0)$ matter?

Given a path $\alpha: I \to X$ from x_0 to x_1 define $\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

<u>Pic</u>



Announcements

- HW due Tues
- Colloquium Friday noon F03-200A
- Class notes and HW1 solutions posted

Outline

- Review of last time
- Properties of the fundamental group
- A word about functors

Review of last time

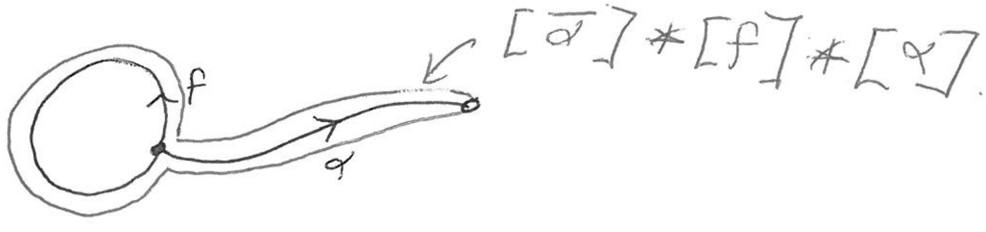
Given a space X and a base point $x_0 \in X$, the fundamental groups of X, denoted $\pi_1(X, x_0)$, is the set of path homotopy classes of loops in X based at x_0 together with the opperation $*$.

$\underline{\text{Ex}}$ $\pi_1(\mathbb{R}^n, \vec{0}) \cong \{1\}$

$\underline{\text{Def}}$ Let α be a path in X from x_0 to X_1.

We define $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$



Thm (52.1) The map $\hat{\alpha}$ is a group isomorphism.

Proof

① Show $\hat{\alpha}$ is a homomorphism

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\bar{\alpha}]*[f]*[\alpha]*[\bar{\alpha}]*[g]*[\alpha]$$
$$= [\bar{\alpha}]*[f]*[e_{x_0}]*[g]*[\alpha]$$
$$= [\bar{\alpha}]*[f]*[g]*[\alpha]$$
$$= [\bar{\alpha}]*[f*g]*[\alpha]$$
$$= \hat{\alpha}([f*g])$$

② To show $\hat{\alpha}$ is a bijection it sufices to show the following maps are the idetity maps

$$\hat{\alpha} \circ \hat{\bar{\alpha}} \quad \text{and} \quad \hat{\bar{\alpha}} \circ \hat{\alpha}$$

$$\hat{\alpha} \circ \hat{\bar{\alpha}}([h]) = \hat{\alpha}([\alpha]*[h]*[\bar{\alpha}])$$
$$= [\bar{\alpha}]*[\alpha]*[h]*[\bar{\alpha}]*[\alpha]$$
$$= [e_{x_1}]*[h]*[e_{x_1}]$$
$$= [h]$$

Hence, $\hat{\alpha} \circ \hat{\bar{\alpha}} = id_{\pi_1(X, x_1)}$

A similar argument shows

$$\hat{\bar{\alpha}} \circ \hat{\alpha} = id_{\pi_1(X, x_0)}$$

Cor (52.2) If X is path connected any $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Note We will often be restricting ourselfs to path-connected top. spaces.

Def A space X is simply connected if it is path connected and $\pi_1(X, x_0) \cong \mathbb{1}$.

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In math we often study classes of objects together with natural maps between those objects called morphisms

Category	Morphisms
Sets	function
Groups	homomorphisms
Top. spaces	Continuous functions
Vect. Spaces	linear maps.

Maps between categories that preserve ~~morphisms are called functors.~~ identity morphisms and composition of morphisms are called functors.

Lemma 52.3 In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Pf Suppose X is simply connected and

$f: I \to X$ and $g: I \to X$ are paths s.t. $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$.

note: $f * \bar{g}$ is a loop based at x_0

By our previous corrollary, $\pi_1(X, x_0) \cong \{1\}$.

Hence $f * \bar{g} \simeq_p e_{x_0}$.

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$$[g] = [e_{x_0}] * [g] = [f] * [\bar{g}] * [g]$$
$$= [f] * [e_{x_1}]$$
$$= [f]$$

Hence, $[g] = [f]$.

Thus $g \simeq_p f$. $\square$

Let $h: X \to Y$ be a continuous map s.t. $h(x_0) = y_0$. Define $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

by $h_*([f]) = [h \circ f]$

<u>Claim</u> h_* is a homomorphism.

Let $[f], [g] \in \pi_1(X, x_0)$.

$$
\begin{aligned}
h_*([f] * [g]) &= h_*([f * g]) \\
&= [h \circ (f * g)] \\
&= [(h \circ f) * (h \circ g)] \\
&= [h \circ f] * [h \circ g] \\
&= h_*([f]) * h_*([g]). \quad \square
\end{aligned}
$$