Review | Given $h: X \to Y$ is a continuous map s.t. $h(x_0) = h(y_0)$,

The map $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $h_*([f]) = [h \circ f]$ is

① well-defined

② a group homomorphism

Th^m | (52.5) If $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ are continuous maps then $(k \circ h)_* = k_* \circ h_*$. Moreover, if $i: (X, x_0) \to (X, x_0)$ is the identity map, then i_* is ~~an isomorphism.~~ the identity homomorphism.

Pf | Examine $(k \circ h)_*([f])$ for $[f] \in \pi_1(X, x_0)$

$$= \cancel{[(k \circ h)([f]}\; [k \circ h \circ f]$$
$$= k_*([h \circ f])$$
$$= k_*(h_*([f]))$$
$$= k_* \circ h_*([f]). \quad \square$$

$$i_*([f]) = [i \circ f] = [f]. \quad \square$$

Note: Th^m 52.5 verifies exactly the two additional criteria that garantee that π_1 is a functor!!!

Covering maps

Def Let $p: E \to B$ be a continuous onto map. A open set $U \subset B$ is said to be <u>evenly covered</u> if $p^{-1}(U)$ can be written as the disjiont union of disjiont open sets V_α in E s.t

$\forall \alpha \quad p|_{V_\alpha}: V_\alpha \to U$ is a homeomorphism.

$\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into slices.

Trivial example

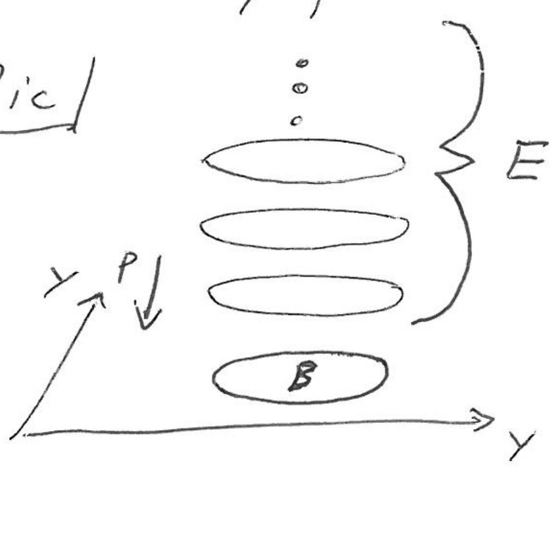
$B =$ unit disk in xy-plane

$E = \bigcup_{i=1}^{\infty} B \times \{i\} \subset \mathbb{R}^3$

Define $p: E \to B$ by $p(x, y, z) = (x, y)$.

In this case <u>all</u> open sets in B are evenly covered by p.

Pic



Def | Let $p: E \to B$ be a continuous onto map. If every point $b \in B$ has a nbh U_b s.t. U_b is evenly covered by p, then p is a covering map and E is a covering space of B.

Observations

① Let $p: E \to B$ be a covering map.
What ~~topology~~ what is the subspace topology on $p^{-1}(b)$ for any point $b \in B$.
The discrete topology.

Why? $\forall b \in B$ $\exists U_b$ s.t. U_b is evenly covered by p. By chosing one slice of $p^{-1}(U_b)$ and intersecting it with $p^{-1}(b)$, we see $p^{-1}(b)$ has the discrete topology.

② If $p: E \to B$ is a covering map, then p is an open map.

Pf | Let $A \subset E$ be open. WTS $p(A)$ is open.
Let $b \in p(A)$. Since p is a covering map $\exists$ a nbh U_b of b s.t. U_b is evenly covered by p.
Let $\{V_\alpha\}$ be the slices of $p^{-1}(U_b)$. Pick α s.t. $V_\alpha \cap p^{-1}(b) \cap A \neq \emptyset$
$\exists y \in V_\alpha$ s.t. $y \in A$ and $p(y) = b$.
~~Fix α. There exists $y \in V_\alpha$ s.t. $p(y) = b$~~
Note $V_\alpha \cap A$ is open in V_α and open in A.
Since $p|_{V_\alpha}$ is a homeomorphism $p|_{V_\alpha}(V_\alpha \cap A)$ is an open nbh of b in $p(A)$. □

Th^m (53.1) The map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map.

<u>Pf</u>| The continuity and surjectivity of p follows from elementary properties of sine and cosine.

Define
$$\mathbb{R}^2 \supset \mathbb{H}^{\uparrow} = \{(x,y) \mid y > 0\}$$
$$\mathbb{H}^{\downarrow} = \{(x,y) \mid y < 0\}$$
$$\mathbb{H}^{\rightarrow} = \{(x,y) \mid x > 0\}$$
$$\mathbb{H}^{\leftarrow} = \{(x,y) \mid x < 0\}$$

Let $x \in S^1$. Note $x \in S^1 \cap \mathbb{H}^*$ for some $* \in \{\uparrow, \downarrow, \rightarrow, \leftarrow\}$.

WLOG assume $x \in S^1 \cap \mathbb{H}^{\rightarrow}$.

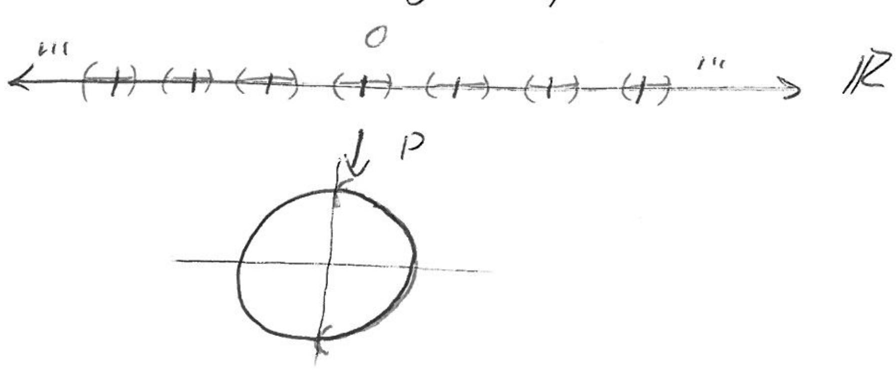
Let $V_i = (i - 1/4, i + 1/4)$.

Note that $p^{-1}(S^1 \cap \mathbb{H}^{\rightarrow}) = \bigcup_{i=-\infty}^{\infty} V_i$ and

$p|_{V_i} : V_i \to S^1 \cap \mathbb{H}^{\rightarrow}$ is a homeomorphism

Hence p is a covering map. $\square$

<u>Pic</u>|



Thm (53.2) Let $p: E \to B$ be a covering map. Let $B_0 \subset B$ be a subspace. Let $E_0 = p^{-1}(B_0)$, then $p|_{E_0}: E_0 \longrightarrow B_0$ is a covering map.

Pf Since p is continuous and surjective, then $p|_{E_0}$ is continuous and surjective. Let $b \in B_0 \subset B$.

Since p is a covering map, $\exists\, U_b$ a nbh of b s.t. U_b is evenly covered by p. Let $\{V_\alpha\}$ be the partition of $p^{-1}(U_b)$ into slices.

Note $p|_{E_0}^{-1}(U_b \cap B_0) = \{V_\alpha \cap E_0\}$ and

since $p|_{V_\alpha}: V_\alpha \to U_b$ is a homeomorphism

$$(p|_{E_0})|_{V_\alpha \cap E_0} : V_\alpha \cap E_0 \to U_b \cap B_0 \text{ is a}$$

homeomorphism. Thus $U_b \cap B_0$ is a nbh of $b \in B_0$. That is evenly covered by $p|_{E_0}$.

Hence $p|_{E_0}$ is a covering map.

$\underline{\text{Th}^m\ (53.3)}$ If $p : E \to B$ and $p' : E' \to B'$ are covering spaces, then $p \times p' : E \times E' \to B \times B'$ is a covering space.

$\underline{\text{Pf}}$ The fact that $p \times p'$ is continuous and surjective follows from basic properties of product maps.

Let $(b, b') \in B \times B'$.

Since p is a covering map $\exists\, U_b$ a nbh of b s.t. U_b is evenly covered by p. S.t. $p^{-1}(U_b) = \{V_\alpha\}$.

Simillary $\exists\, U_{b'}$ s.t. $U_{b'}$ is evenly covered by p'. s.t. $(p')^{-1}(U_{b'}) = \{V'_{\alpha'}\}$.

$U_b \times U_{b'}$ is an open nbh of (b, b')

Note that $(p \times p')^{-1}(U_b \times U_{b'}) = \cancel{\{V_\alpha\} \times \{V'_{\alpha'}\}}$

$$= \bigcup_{\substack{\alpha \\ \alpha'}} V_\alpha \times V'_{\alpha'}$$

Similarly $(p \times p')\big|_{V_\alpha \times V'_{\alpha'}} : V_\alpha \times V'_{\alpha'} \to U_b \times U_{b'}$

is a homeomorphism for all α and α'.

Thus $p \times p'$ is a covering map.

Example Recall $p: \mathbb{R} \to S^1$ via

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

is a covering map.

By thm (53.3) $p \times p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$

is a covering map.