Announcements

- HW due a week from today
- Colloquim speaker on knot theory Friday 3pm F03-200A

Outline

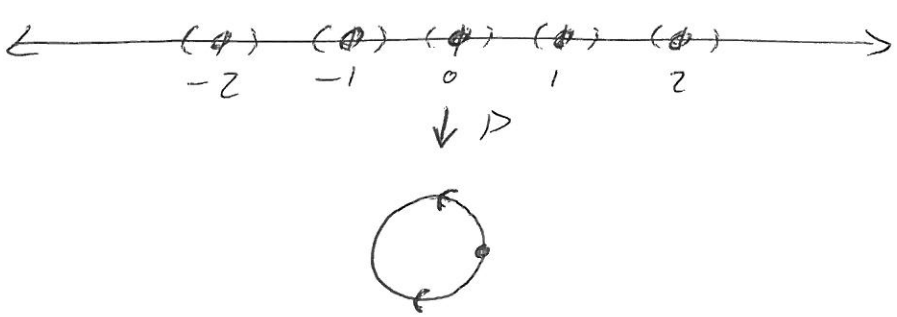
- Review from last time
- lifts of maps
- $\pi_1(S^1, x_0) \cong \mathbb{Z}$.

Review

Last time we showed $p: \mathbb{R} \to S^1$ via

$p(x) = (\cos(2\pi x), \sin(2\pi x))$ is

a covering map.



Lifts | let $p: E \to B$ be a covering map. If $f: X \to B$ is a continuous function, a lifting of f is a map $\hat{f}: X \to E$ s.t. $f = p \circ \hat{f}$.

Example

$p : \mathbb{R} \to S^1$ via $p(x) = (\cos(2\pi x), \sin(2\pi x))$

$f : [0,1] \to S^1$ via $f(x) = (\cos(\pi x), \sin(\pi x))$

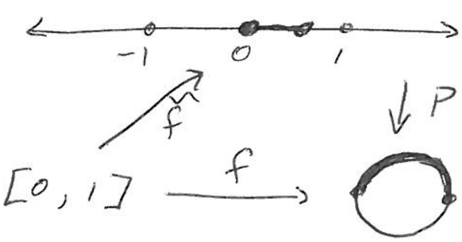
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

What is $\tilde{f}$? (If it exists)

$\tilde{f} : [0,1] \to \mathbb{R}$

$\tilde{f}(x) = \frac{1}{2}x \leftarrow$ not unique!

$\tilde{f}(x) = \frac{1}{2}x + 5$



Lemma (54.1) Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ begining at b_0 has a unique lift to a path $\tilde{f} : [0,1] \to E$ beginning at e_0.

<u>Pf</u>| By def. of a covering map, for every $b \in B$ $\exists\, U_b$ a nbh of b s.t. U_b is evenly covered by p. Hence $\{U_b\}_{b \in B}$ is an open cover for B. Since $[0,1]$ is compact and f is continuous, then there exists a finite subcover $\{U_i\}_{i=1}^{n}$ that covers $f([0,1])$.

By the Lebesgue number theorem, there exist $0 \leq s_0 \leq s_1 \cdots \leq s_n \leq 1$ s.t. $f([s_i, s_{i+1}])$ is entirely contained in some U_j.

Step 1 | Define $\check{f}(0) = e_0$.

Step 2 | Assumeing $\check{f}$ is defined for $0 \leq s \leq s_i$, we define $\check{f}$ on $[s_i, s_{i+1}]$ as follows: The set $f([s_i, s_{i+1}])$ lies in some U_j which is evenly-covered by p. Let $\{V_\alpha\}_{\alpha \in A}$ be a partition of $p^{-1}(U_j)$ into slices. $\check{f}(s_i)$ lies in V_β for some $\beta \in A$. Define $\check{f}(s)$ for $s \in [s_i, s_{i+1}]$ by

$$\check{f}(s) = \underbrace{(p|_{V_\beta})^{-1}(f(s))}_{\text{continuous on } [s_i, s_{i+1}] \text{ since } p|_{V_\beta} \text{ is a homeomorphism.}}$$

Hence, $\check{f}(s)$ is continuous on $[0, s_{i+1}]$ by the pasting lemma. Thus, we define $\check{f}$ on all of $[0,1]$ inductively and the continuity of $\check{f}$ is garanteed by the pasting lemma.

Note $p \circ \check{f}(s) = p \circ ((p|_{V_\beta})^{-1}(f(s)))$ (for some $\beta \in A$)
$= f(s)$.

Thus $\check{f}$ is a lift of f.

Next, we show $\tilde{f}$ is the <u>unique</u> lift.

Suppose $\tilde{\tilde{f}}: [0,1] \to E$ is another lifting of f s.t. $\tilde{\tilde{f}}(0) = e_0$.

Hence $\tilde{\tilde{f}}(0) = e_0 = \tilde{f}(0)$.

Suppose, $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for $0 \leq s \leq s_i$.

WTS $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$.

Let $f([s_i, s_{i+1}]) \subset U_j$ for some j.

~~Let~~ Then $p^{-1}(U_j)$ is partitioned into $\{V_\alpha\}_{\alpha \in A}$.

Let $\tilde{\tilde{f}}(s_i) = \tilde{f}(s_i) \in V_\beta$. Recall $\tilde{f}(s) = (p|_{V_\beta})^{-1}(f(s))$.

Since $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is connected and the V_α are disjoint then $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is entirely contained in some fixed V_γ. Since $\tilde{\tilde{f}}(s_i) \in V_\beta$ then $\tilde{\tilde{f}}([s_i, s_{i+1}]) \subset V_\beta$.

Let $t \in [s_i, s_{i+1}]$ $\tilde{\tilde{f}}(t) \in V_\beta$ s.t. $p \circ \tilde{\tilde{f}}(t) = f(t)$.

Hence $\tilde{\tilde{f}}(t) = (p|_{V_\beta})^{-1}(f(t)) = \tilde{f}(t)$.

Thus, $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for $0 \leq s \leq s_{i+1}$ by pasting lemma.

By induction $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for all $s \in [0,1]$.

Hence, $\tilde{f}(s)$ is a <u>unique</u> lift. □

Announcements

- HW due on Tuesday
- Colloquium speaker on Knots & links 3pm Friday F03 - 200A.

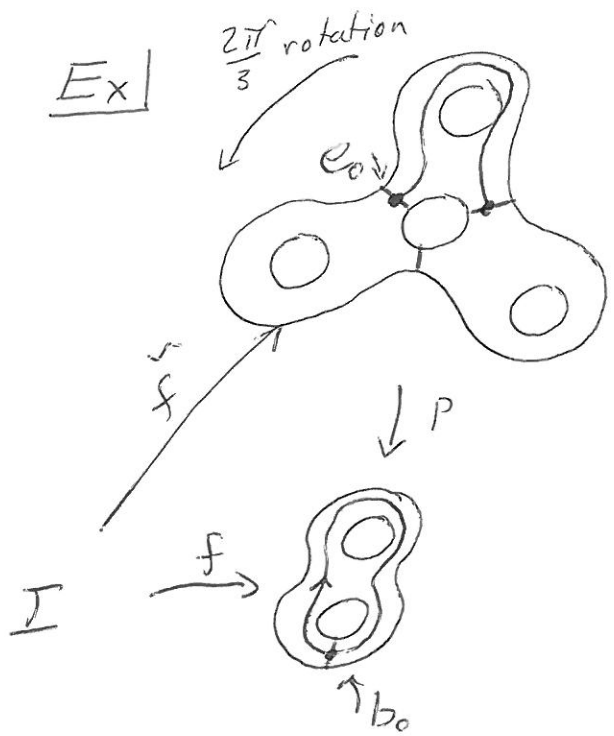
Outline

- path homotopy lifting theorem.

Last time

Lemma (54.1) If $p: E \to B$ is a covering map s.t. $p(e_0) = b_0$ and $f: I \to B$ is a path s.t. $f(0) = b_0$, then there exists a unique lift of f, denoted $\tilde{f}: I \to E$ s.t. $\tilde{f}(0) = e_0$.

Ex



Lemma (54.2)

Let $p: E \to B$ be a covering map s.t. $p(e_0) = b_0$.

Let $F: I \times I \to B$ be a continuous map s.t. $F(0,0) = b_0$.

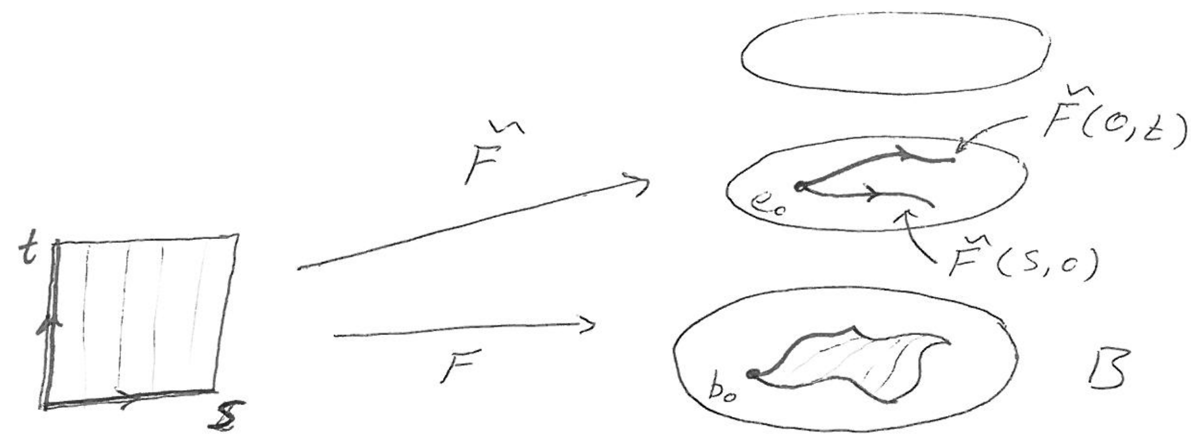
There is a <u>unique</u> lifting of F to a continuous map $\tilde{F}: I \times I \to E$ s.t. $\tilde{F}(0,0) = e_0$

Furthermore, if F is a path-homotopy, then $\tilde{F}$ is a path homotopy.

<u>Proof</u> Define $\tilde{F}(0,0) = e_0$

Use lemma 54.1 to ~~extend~~ lift $F(s,0)$ to a unique path $\tilde{F}(s,0)$.

Similarly, lift $F(0,t)$ to a unique path $\tilde{F}(0,t)$.



Since p is a covering space, for every $b \in B$ there exists U_b a nbh of b s.t. U_b is evenly covered by p. Hence, $\{U_b\}_{b \in B}$ is an open cover of B.

Since $F(I)$ is compact, let $\{U_\ell\}_{\ell=1}^{k}$ be a subcover that covers $F(I)$. By the Lebesgue number lemma, there exist subdivitions $s_0 < s_1 < \cdots < s_m$ and $t_0 < t_1 < \cdots < t_n$ s.t.

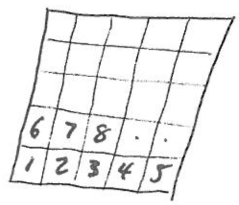
Each rectangle $I_i \times J_j = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$
has the property that there exists $1 \leq l \leq k$ s.t.

$$F(I_i \times J_j) \subset U_l.$$

We will define $\check{F}$ inductively on

$$I_1 \times J_1 \cup I_2 \times J_1 \cup \cdots \cup I_{m-1} \times J_1 \cup I_1 \times J_2 \cup I_2 \times J_2 \cup \cdots \cup I_{m-1} \times J_{n-1}$$

Pic



Assume $\check{F}$ is defined and continuous on $I \times \{0\}$, $\{0\} \times I$ and all rectangles previous to $I_{i_0} \times J_{j_0}$. Call this A. WTS $\check{F}$ is defined and continuous all all this union $I_{i_0} \times J_{j_0}$.
Let U_l be a set s.t. $F(I_{i_0} \times J_{j_0}) \subset U_l$
Since p evenly covers U_l, let $p^{-1}(U_l)$ be partitioned into disjiont open slices $\{V_\alpha\}_{\alpha \in A}$.
Let C be the union of the left and bottom edges of $I_{i_0} \times J_{j_0}$.
Since $\check{F}$ is defined and continuous on C, then $\check{F}(C)$ is connected. Hence, $\check{F}(C) \subset V_\beta$ for some $\beta \in A$.
Since $\check{F}$ is a lifting of F ~~when~~ on the domain it is defined on, then

$$(p|_{V_\beta}) \circ (\check{F}(x)) = F(x) \text{ for } x \in C.$$

$$\text{So} \quad \check{F}(x) = (p|_{V_\beta})^{-1}(F(x)) \text{ for } x \in C.$$

Define $\check{F}(x) = (p|_{V_\beta})^{-1}(F(x))$ for $x \in I_{i_0} \times J_{j_0}$
By the pasting lemma $\check{F}(x)$ is continuous on $I_{i_0} \times J_{j_0} \cup A$.
~~union the previous domain.~~

Similarly, $p \circ (\tilde{F}(x)) = p \circ ((p|_{V_\beta})^{-1}(F(x)))$ (for some $\beta \in A$)

$= F(x)$. for all $x \in A \cup I_{i_0} \times J_{j_0}$.

Hence, $\tilde{F}(x)$ is a lift of $F(x)$.

By induction $\tilde{F}(x)$ is a lift of $F(x)$ for all $x \in I \times I$.

A similar argument shows $\tilde{F}$ is unique.

Suppose F is a path homotopy.

Note $F(I \times \{0\})$ and $F(I \times \{1\})$ are both single points b_0 and b_1 respectively.

Hence $\tilde{F}(I \times \{0\}) \subset p^{-1}(b_0)$ ~~and~~ and $\tilde{F}(I \times \{1\}) \subset p^{-1}(b_1)$.

Since $\tilde{F}$ is continuous and $I \times \{t\}$ is connected, then

$\tilde{F}(I \times \{0\})$ is a point in $p^{-1}(b_0)$ (In particular e_0)

and $\tilde{F}(I \times \{1\})$ is a point in $p^{-1}(b_1)$.

Thus $\tilde{F}$ is a path homotopy. $\square$