555 Lec 3

Announcements

- Class cancelled on Thursday
- HW up on web, due Tuesday.

Review

Def If $U \subset \mathbb{R}^n$ is open, $f: U \to \mathbb{R}^m$ is <u>smooth</u> if f has continuous partial derivatives of all orders. (If U is not open, f is smooth if it can be extended to a open subset of $\mathbb{R}^n$)

Def Given $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ $f: X \to Y$ is a diffeomorphism if

1. f is smooth
2. f is a bijection
3. f^{-1} is smooth.

Def Let $X \subset \mathbb{R}^N$. X is a <u>(smooth) k-manifold</u> if for every $x \in X$ there is an open nbh, U_x, of x in X and a diffeomorphism $f: U_x \to V$ where V is an open subset of $\mathbb{R}^k$.

Thm If $X \subset \mathbb{R}^N$ is a smooth k-manifold, then X with the subspace topology is a (topological) manifold.

Pf Recall, we are considering $\mathbb{R}^N$ with the standard topology and $X \subset \mathbb{R}^N$ with the subspace topology τ_X. From 550, $\mathbb{R}^N$ is both 2nd-countable and Hausdorff. From last time, this implies X is both 2nd-countable and Hausdorff.

To show X is locally Euclidean let $x \in X$. By def. of smooth k-manifold $\exists\ U_x \subset X$ and open set containing x s.t. U_x is diffeomorphic to V, an open subset of $\mathbb{R}^k$. However, diffeomorphic implies homeomorphic, so U_x is homeomorphic to V. Hence, X is locally Euclidean. $\square$

Coordinates

Let $X \subset \mathbb{R}^N$ be a k-smooth manifold.

$\forall x \in X \ \exists V_x \subset X$ s.t. V_x is a nbh of x in X and there is a diffeomorphism ~~from~~ $f: U_x \to V_x$ for some open set U_x in $\mathbb{R}^k$.

f is a <u>parametrization</u> of V_x

f^{-1} is a <u>coordinate system</u> on V_x

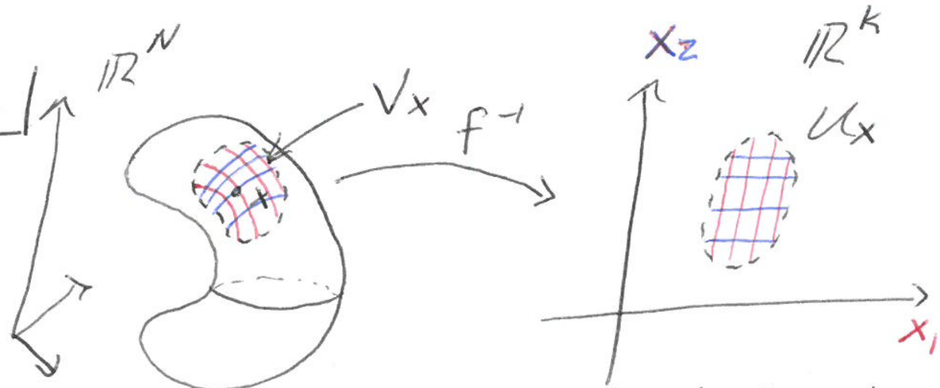
Note $f^{-1}: V_x \longrightarrow U_x \subset \mathbb{R}^k$

So, $f^{-1} = \langle x_1, x_2, \ldots, x_k \rangle$ where

$$x_i : V_x \longrightarrow \mathbb{R}$$

Each x_i is a <u>coordinate function</u>.

Pic



We can implicitly identify V_x with U_x by identifying $v \in V_x$ with $\langle x_1(v), x_2(v), \ldots, x_k(v) \rangle$.

Notation If X is a k-dim'l smooth manifold, we say $\dim(X) = k$.

Thm If X and Y are smooth manifolds, then $X \times Y$ is a smooth manifold and $\dim(X \times Y) = \dim(X) + \dim(Y)$

Pf Suppose X is a k-dim'l manifold in $\mathbb{R}^N$ and Y is a l-dim'l manifold in $\mathbb{R}^M$. $X \times Y$ is a subset of $\mathbb{R}^N \times \mathbb{R}^M = \mathbb{R}^{N+M}$.

Let $(x, y) \in X \times Y$.

Since $x \in X$ $\exists$ a local parametrization $\phi : W \to X$ around x (where $W \subset \mathbb{R}^k$ is open)

Since $y \in Y$ $\exists$ a local parametrization $\psi : U \to Y$ around y (where $U \subset \mathbb{R}^l$ is open)

Define $\phi \times \psi : W \times U \to X \times Y$ s.t.

$$\phi \times \psi (w, u) = (\phi(w), \psi(u))$$

Note $W \times U$ is open in $\mathbb{R}^{k+l}$

- Since the product of smooth functions is smooth, $\phi \times \psi$ is smooth.
- Since ϕ and ψ are invertable, then $\phi \times \psi$ is invertable with inverse $\phi^{-1} \times \psi^{-1}(x, y) = (\phi^{-1}(x), \psi^{-1}(y))$.
- Showing $\phi^{-1} \times \psi^{-1}$ is smooth is a bit more subbtle, but still true.
- Hence $X \times Y$ is a smooth manifold in $\mathbb{R}^{N+M}$ and $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Def Suppose X and Z are smooth manifolds in $\mathbb{R}^N$ s.t. $Z \subset X$, then we say Z is a <u>submanifold</u> of X.

Note: Any open subset of a manifold is a manifold.

Derivatives & Tangents

Given a ~~map~~ smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, denote the derivative of f at x by df_x.

Chain rule $d(g \circ f)_x = dg_{f(x)} \circ df_x$

In other words, from the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V \arrow[r, "g"] & \mathbb{R}^l \end{tikzcd}$$

we get

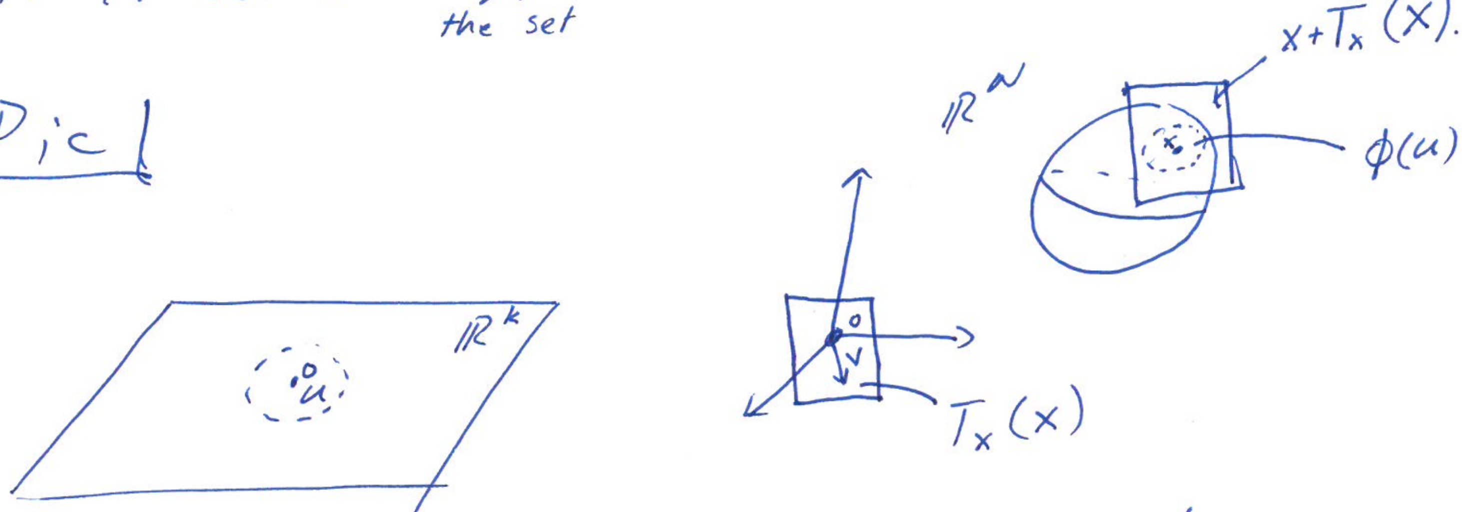
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "df_x"] \arrow[rr, bend right, "d(g \circ f)_x"'] & \mathbb{R}^m \arrow[r, "dg_{f(x)}"] & \mathbb{R}^l \end{tikzcd}$$

Let X be a smooth k-manifold in $\mathbb{R}^N$ and let $\phi: U \to X$ be a local parametrization at the point x. For convience, assume $\phi(0) = x$.

<u>Def</u> The <u>tangent space</u> of X at x is the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$, and we denote it by $T_x(X)$.

Note: The best k-dim'l flat approximation to X at x is the set $x + T_x(X) \subset \mathbb{R}^N$.

<u>Pic</u>



Note $v \in T_x(X)$ is called a <u>tangent vector</u>.

Question: Is tangent ~~vector we~~ space well-defined?