Differential topology Lec. 6.

<u>Inverse function Th^m</u>| Let X and Y be smooth manifolds and suppose $f: X \to Y$ is a smooth map s.t. df_x is an isomorphism. Then f is a local diffeomorphism at x.

<u>Pf</u>| Too much analysis. I sent a link for those interested.

Q: What is the best local behavior we can ask of a smooth map $f: X \to Y$ if $\dim(X) < \dim(Y)$?

A: df_x is one-to-one.

<u>Def</u>| If df_x is one-to-one at x we say f is an <u>immersion at x</u>.

Example: $f: \mathbb{R}^k \to \mathbb{R}^\ell$ with $k \leq \ell$ s.t.

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$$ is an immersion on its entire domain.

<u>Local Immersion Th^m</u>| Suppose $f: X \to Y$ is an immersion at x and $y = f(x)$. Then there exists local coordinates around x and y s.t.

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0).$$

Pf | As in our construction of df_x, we can choose parameterizations about x and y that yield the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{U} \arrow[u, "\phi"] \arrow[r, "g"'] \arrow[ur, phantom, "\#"] & V \arrow[u, "\gamma"'] \end{tikzcd} \qquad \begin{array}{l} \phi(0) = x \\ \gamma(0) = y \end{array}$$

$\mathcal{U} \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$ and $df_x = d\gamma_0 \circ dg_0 \circ (d\phi_0)^{-1}$

Since df_x is one-to-one, by assumption and $d\gamma_0$ and $d\phi_0$ are isomorphisms, then dg_0 is one-to-one.

From linear algebra, after a change of basis in $\mathbb{R}^l$ we can assume $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is an $l \times k$ matrix of the form

$$dg_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ or } \begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

Define $G : \mathcal{U} \times \mathbb{R}^{l-k} \to \mathbb{R}^l$ via $G(x,z) = g(x) + (0, \dots, 0, z)$.

$$dG_0 : \mathbb{R}^l \to \mathbb{R}^l$$

$$dG_0 = \begin{pmatrix} I_k & 0 \\ 0 & I_{l-k} \end{pmatrix} = I_l.$$

By the inverse function theorem, G is a local diffeomorphism of $\mathbb{R}^l$ at 0. By definition, $\gamma: V \to Y$ is a local diffeomorphism at 0.

Hence ~~$G \circ \gamma$~~ $\gamma \circ G$ is a local diffeomorphism at 0.

In other words, $\gamma \circ G$ is a local parameterization of $y = f(x)$ in Y.

Hence, after shrinking U and V, we get the following commutative diagram

$$\begin{tikzcd} X \arrow[r] & Y \\ U \arrow[u, "\phi"] \arrow[r] & V \arrow[u, "\gamma \circ G"'] \end{tikzcd}$$

$(x_1, \dots, x_k) \to (x_1, \dots, x_k, 0, \dots, 0)$ (canonical immersion)

So, in local coordinates, $f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$. $\square$.

Facts: The image of a local diffeomorphism is a submanifold

Q: Is the image of a local immersion a submanifold

A: No

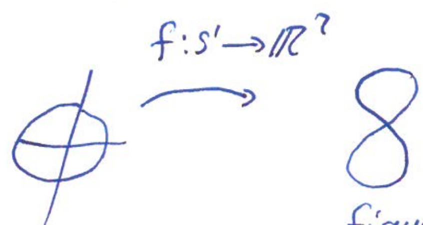


figure 8.

Example/ Immersions can be bad.

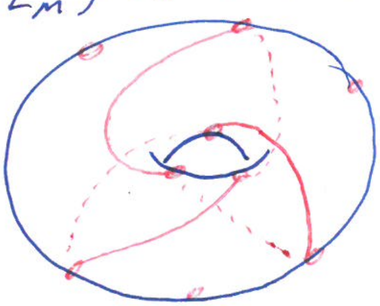
$g: \mathbb{R}^1 \to S^1$ by $g(t) = (\cos(2\pi t), \sin(2\pi t))$ is a local diffeomorphism.

Similarly, $g \times g : \mathbb{R}^2 \to S^1 \times S^1$ is a local diffeomorphism.

Let $L_m \subset \mathbb{R}^2$ be a line through the orgin in $\mathbb{R}^2$ of slope m.

$g \times g|_{L_m} : L_m \to S^1 \times S^1$ is an immersion.

If $m = p/q$ where $(p,q)=1$ and $p,q \in$ ~~#~~ ~~Z~~, and we view $S^1 \times S^1 \subset \mathbb{R}^3$, then $\mathrm{Im}(g \times g|_{L_m})$ is an $L(p,q)$ torus knot.



$L(3,2)$.

If m is irrational $g \times g|_{L_m}$ is one-to-one! More over $\mathrm{Im}(g \times g|_{L_m})$ is dence in $S^1 \times S^1$!

Def| $f: X \to Y$ is *proper* if the pré image of every compact set in Y is compact in X.

Def| If $f: X \to Y$ is an injective, proper immersion, then f is an *embedding*.

Ex| Knot theory is the study of Embeddings of S^1 into $\mathbb{R}^3$ (or S^3).

Thm| An embedding $f: X \to Y$ maps X diffeomorphically onto a submanifold of Y.

Pf|