Lec. 7

From last time:

<u>Def</u> A smooth map $f: X \to Y$ is an <u>imersion</u> at x if df_x is one-to-one.

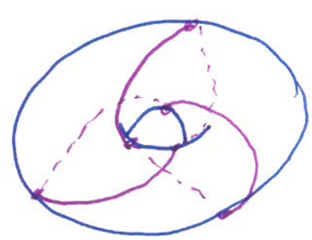
<u>Examples</u> $g: \mathbb{R} \to S^1$ by $g(t) = (\cos(2\pi t), \sin(2\pi t))$

$g \times g: \mathbb{R}^2 \to S^1 \times S^1$

let L_m be a line of slope m in $\mathbb{R}^2$.

$g \times g|_{L_m}: \mathbb{R} \to S^1 \times S^1$ is provabley an imersion

If $m \in \mathbb{Q}$, then $\mathrm{im}(g \times g|_{L_m})$ is nice



If $m \in \mathbb{R} - \mathbb{Q}$, then $\mathrm{im}(g \times g|_{L_m})$ is dence in $S^1 \times S^1$ (Very bad)

It would be ~~th~~ nice if the image of immersions were submanifolds, but they are not always.

Def $f: X \to Y$ is *proper* if the pre image of every compact set in Y is compact in X.

Def If $f: X \to Y$ is an injective, proper immersion, then f is an *embedding*.

Ex Knot theory is the study of Embeddings of S^1 into $\mathbb{R}^3$ (or S^3).

Thm An embedding $f: X \to Y$ maps X diffeomorphically onto a submanifold of Y.

Pf

Def A smooth map $f: X \to Y$ is a submersion at x if df_x is onto.

(Best "local condition" we can hope for if $\dim(Y) \le \dim(X)$)

Ex $f: \mathbb{R}^k \to \mathbb{R}^l$ if $l \le k$

via $f((x_1, x_2, \dots, x_k)) = (x_1, \dots, x_l)$.

is the "canonical submersion"

Th^m Suppose $f: X \to Y$ is a submersion at $x \in X$ and $f(x) = y$. Then there exists local coordinants around x and y s.t.

$$f((x_1, \dots, x_k)) = (x_1, \dots, x_l).$$

(ie., locally, f is equivalent to the canonical submersion)

Pf (Similar to last time)

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \subset \mathbb{R}^k \arrow[u, "\phi"] \arrow[r, "g"] & V \subset \mathbb{R}^l \arrow[u, "\gamma"'] \end{tikzcd}$$

s.t. $\phi(0) = \cancel{\gamma(0)}\ x$
$\gamma(0) = y = f(x)$.

Since $df_x = d\gamma_0 \circ dg_0 \circ (d\phi_0)^{-1}$ and $\cancel{dg_0}$ df_x is onto, then dg_0 must be onto.

So $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is an onto linear map.
Hence we can choose a basis for $\mathbb{R}^k$ s.t.

$$dg_0 = \left[I_l \mid 0 \right]$$

Define $G : U \to \mathbb{R}^k$ by

$$G((x_1, \dots, x_k)) = (g((x_1, \dots, x_k)), x_{l+1}, \dots, x_k)$$

Then

$$dG_0 = \underbrace{\left[\begin{array}{c|c} I_l & 0 \\ \hline 0 & I_{k-l} \end{array}\right]}_{k \times k} = I_k$$

By Inverse function theorem G is a local diffeomorphism.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y & & X \arrow[r] & Y \\
U \arrow[u, "\phi"] \arrow[r, "G"'] & U' \arrow[r, "\text{canonical submersion}"'] & V \arrow[ul, phantom, ""] \arrow[u, "\gamma"'] \arrow[r, phantom, "\longrightarrow"] & U' \arrow[u, "\phi \circ G^{-1}"] \arrow[r, "\text{canonical}"'] & V \arrow[u, leftrightarrow]
\end{tikzcd}$$

After carefully choosing U', $\phi \circ G^{-1}$ is a local parameterization at $x \in X$.
and $f((x_1, \dots, x_k)) = (x_1, \dots, x_l)$. $\square$

Def| Given a smooth map $f: X \to Y$, $y \in Y$ is a regular value if for every $x \in f^{-1}(y)$ df_x is surjective.

Thm| (Big!) If y is a regular value of a smooth map $f: X \to Y$, then $f^{-1}(y)$ is a submanifold of X with $\dim(f^{-1}(y)) = \dim(X) - \dim(Y)$.

Ex| $f: \mathbb{R}^k \to \mathbb{R}$

$$f((x_1, \dots, x_k)) = x_1^2 + x_2^2 + \dots + x_n^2$$

$$df_{(x_1, \dots, x_n)} = \underbrace{\begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & \cdots \end{bmatrix}}_{1 \times k}$$

So $df_{(x_1, \dots, x_n)}$ is onto whenever $(x_1, \dots, x_n) \neq (0, \dots, 0)$.

In particular $1 \in \mathbb{R}$ is a regular value. So S^{k-1} is a smooth $k-1$ manifold.