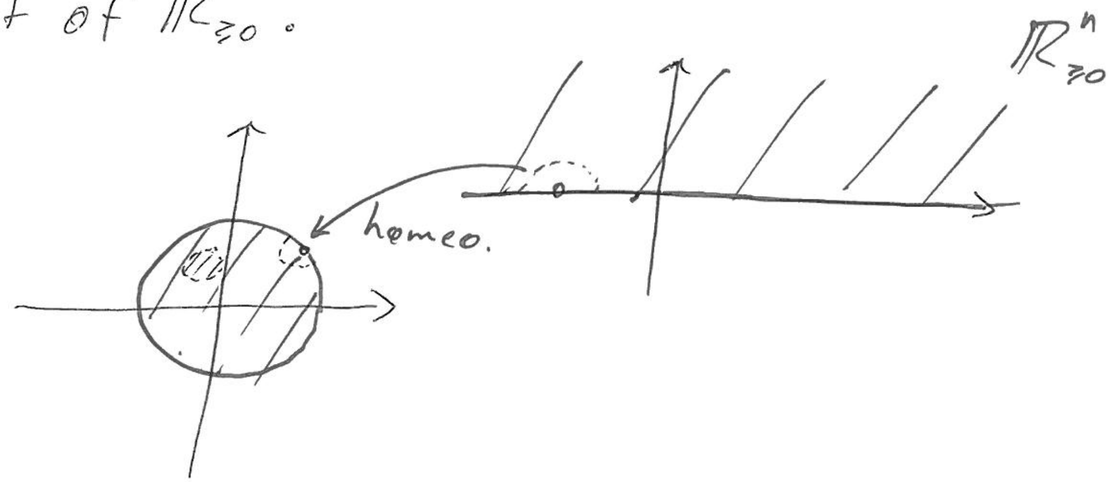
Knot theory Surface & 3-manifold Boot Camp.

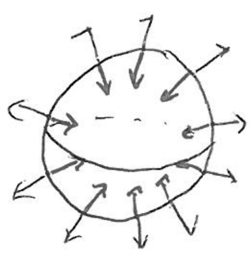
Let $\mathbb{R}^n_{\geq 0} = \{(x, y, z) \in \mathbb{R}^n \mid z \geq 0\}$

<u>Def</u>| A surface is a **Hausdorff** 2nd-Countable topological **space** that is locally homeomorphic to an open subset of $\mathbb{R}^n_{\geq 0}$.

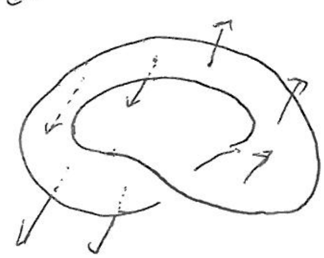
<u>Ex</u>|



<u>Def</u>| A surface is orientable, if there is a consistent choice of normal vector



S^2 is orientable



Möbius band is not orientable.

<u>~~Thm~~ / Def</u>| A surface is <u>closed</u> if it is compact and has empty boundary.

<u>Ex</u>| S^2 is closed $\qquad$ $\mathbb{R}^2$ is not closed.

D^2 is not closed

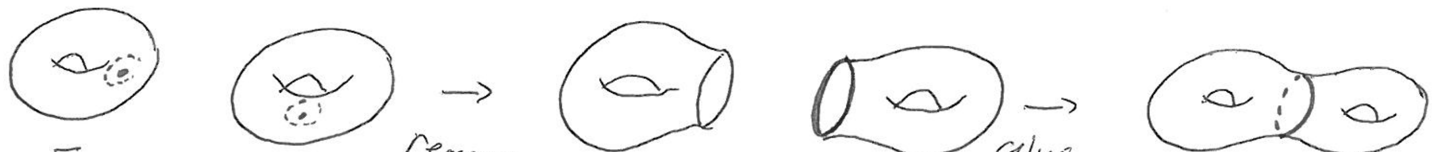
Thm (Classification of Closed Orientable Surfaces)

If F is a closed orientable surface, it is homeomorphic to one of the following

$S^1 \times S^1 \# S^1 \times S^1$ $\quad$ $S^1 \times S^1 \# S^1 \times S^1 \# S^1 \times S^1$

S^2, $g=0$ $\qquad$ $S^1 \times S^1 = \text{torus}$, $g=1$ $\qquad$ $g=2$ $\qquad$ $g=3$ $\qquad \ldots$

Def Connected sum of surfaces



F $\qquad$ H $\qquad$ remove ε-nbhs of a point in Each $\qquad$ glue together along resulting boundary components $\qquad$ $F \# H$

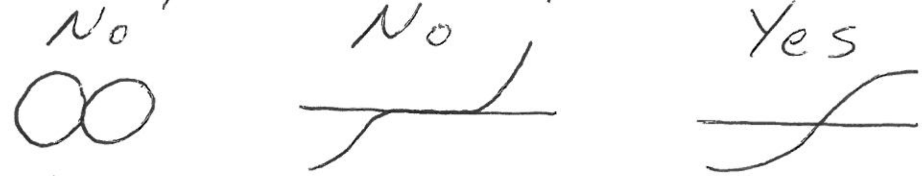
Def The genus of F is the number of tori in a maximal decomposition as a connected sum.

Def Given a decomposition of a surface into ~~points edges~~ vertices, edges and disk faces

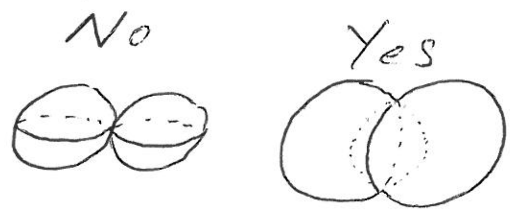
$$\chi(F) = \underset{\#\text{ of vert}}{V} - \underset{\#\text{ of edges}}{E} + \underset{\#\text{ of faces}}{F}.$$

Transversality results.

Thm | Let α and β be simple closed curves embedded in a surface F. After an isotopy of α supported in an arbitrarily small ε-nbh of α, α meets β transversely in a collection of points.



Thm | Let F and G be closed surfaces embedded in a 3-manifold M. After an isotopy of F supported in an arbitrarily small nbh of F, F meets G transversly in a collection of simple closed curves.



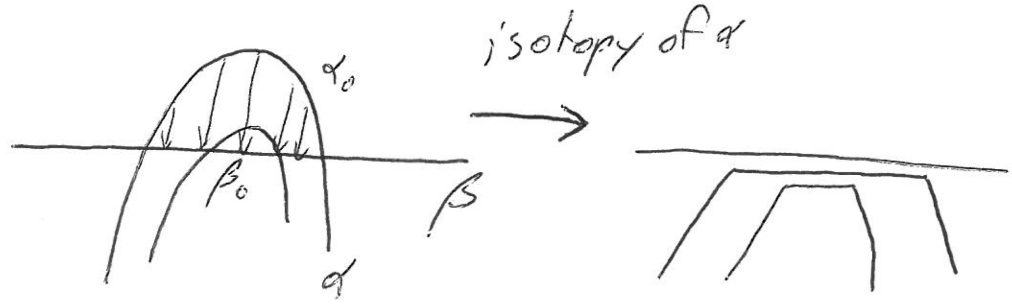
Generally | If F is a closed n-manifold and G is a closed m-manifold embedded in a k-manifold, then there exists an isotopy of F supported in an ε-nbh of F s.t. F meets G transversly in a collection of $n+m-k$ closed manifolds.

Think subspaces of $\mathbb{R}^k$.

Th^{m} | Let α and β be simple closed curves embedded in a surface F such that $\alpha \perp \beta$. If α has been isotoped to minimize $|\alpha \cap \beta|$, then $\alpha \cup \beta$ has no bigons in F.

Def | Given simple closed curves α and β in F have a bigon if there is a sub arc of α, called α_0 and a sub arc of β, called β_0 s.t. $\alpha_0 \cap \beta = \partial \alpha_0$ and $\alpha_0 \cup \beta_0$ bound a disk in F.

Ex

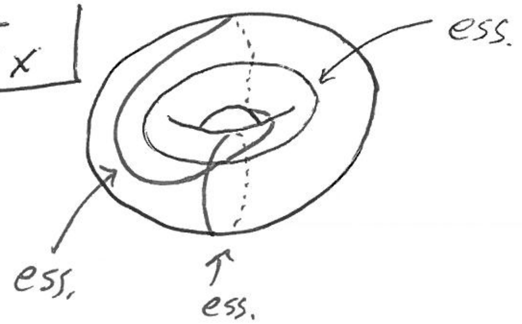


Review

- Transversality
- bigon condition for minimal intersection of s.c.c.

Def | A simple closed curve embedded in a closed surface is <u>essential</u> if it does not bound a disk in the surface.

Ex |



A simple closed curve $f: S^1 \to F$ is essential iff $f_*: \pi_1(S^1, x_0) \to \pi_1(F, f(x_0))$ is injective.

(Note: This is a stronger version of a homotopy result proved in 550B).

Transition to surfaces in 3-manifolds |

Transversality: If F and H are closed surfaces embedded in a 3-manifold M, then after a perterbation of F, $F \cap H$ is a collection of disjiont simple closed curves (possibly empty).

Given two surfaces $f_1 : F \to M$ and $f_2 : F \to M$ embedded in a 3-manifold M, we ussually consider f_1 is equivalent to f_2 if there exists an isotopy taking f_1 to f_2.

$H : F \times I \to M$ s.t. $H(F \times \{t\})$ is an embedding for every t.

Let K be a knot in a 3-manifold M and suppose $f_1 : F \to M$ and $f_2 : F \to M$ are embeddings of surfaces that are <u>transverse to K</u>, We say f_1 is transversly isotopic to f_2 if there exists an isotopy $H : F \times I \to M$ from f_1 to f_2 s.t. $H(F \times \{t\})$ is transverse to K for all t.

Example:

Let $B^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

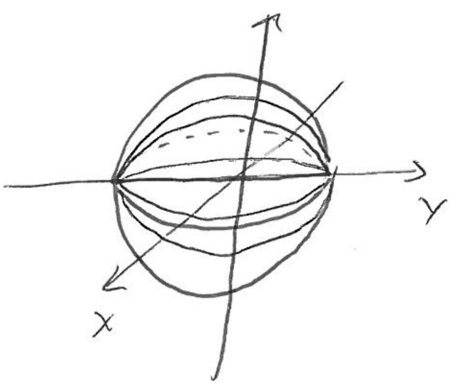
Let $H_1 = \{(x,y,z) \mid x^2+y^2+z^2=1 \text{ and } z \geq 1\}$

$H_2 = \{(x,y,z) \mid x^2+y^2+z^2=1 \text{ and } z \leq 1\}$

Claim | There is an isotopy taking H_1 to H_2 and fixing ∂H_i.

Pf | $H : D^2 \times I \to B^3$

$$H((x,y),t) = \cancel{(1-t)} \langle x, y, (1-t)\sqrt{1-x^2-y^2} \rangle$$
$$\cancel{t} + \langle x, y, -t\sqrt{1-x^2-y^2} \rangle$$



Th^m | If α is a s.c.c. embedded in ∂B^3 then α bounds two disks H_1 and H_2 in ∂B^3 (Jordon Curve th^m) and H_1 is isotopic to H_2 via an isotopy that fixes ∂H_i.

Pf | Similar to ~~s.c.c~~ ~~bigon case~~ s.c.c. case.

Def

An embedded surface $F \subset M$ is <u>incompressible</u> if it is not <u>compressible</u>.

A surface F is compressible if there exists an embedded disk D in M st. $D \cap F = \partial D$ and ∂D is an essential simple closed curve in F.

(loop theorem)

Thm $F \subset M$ is incompressible iff the induced homomorphism corresponding to the inclusion map i is one-to-one $i_* : \pi_1(F, x_0) \to \pi_1(M, x_0)$

Ex Given a satellite knot K, the companion torus is incompressible.

Essentially by definition since the companion must be knoted and the pattern must be "non-trivial".

In fact, a knot is satellite iff it has an incompressible, non-boundary parallel torus in its exterior.

Def A 3-manifold M is <u>reducible</u> if there exists an embedded 2-sphere $F \subset M$ s.t. F does not bound a 3-ball to one side. Otherwise, M is irreducible.

Ex S^3 is irreducible

Thm (Alexanders Thm) If F is a smoothly embedded 2-sphere in S^3 then F bounds a 3-ball to both sides. (3D analogue of Jordan - Curve)

Ex $S^2 \times S^1$ is reducible. Since $S^2 \times \{pt\}$ bounds a $S^2 \times I$ to "both sides".

Ex A knot complement is irreducible.

If not. Let F be a reducing sphere for $S^3 - K$.
By Alexander's thm F bounds a 3-ball to each side.
$S^3 = B_1^3 \cup_F B_2^3$. Since K is connected, $K \subset int(B_1^3)$ or $K \subset int(B_2^3)$. WLOG assume $K \subset int(B_1^3)$, then F bounds the 3-ball B_2^3 to one side. $\square$

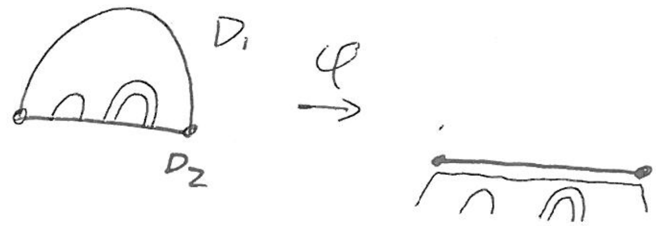
Ex Split link complements are reducible.

Thm\] Suppose F_1 and F_2 are embedded closed incompressible surfaces in an irreducible M. After an isotopy of F_1 and F_2, $F_1 \cap F_2$ is a collection of <u>essential</u> curves in both F_1 and F_2.

Pf\] After a perterbation of F_1 $F_1 \cap F_2$ is a collection of disjiont simple closed curves in F_1 and F_2.

Suppose there exists a curve $\alpha \in F_1 \cap F_2$ that is inessential in F_1. Then α bounds an embedded disk D_1 in F_1. Examine $F_2 \cap D_1$.

If $F_2 \cap D_1 \neq \emptyset$, then rechoose α to be an innermost s.c.c. ~~in~~ of $F_2 \cap D_1$ in D_1. Hence, we can assume $int(D_1) \cap F_2 = \emptyset$. If α is essential in F_2, then F_2 is compressible, a contradiction. Hence, we can assume α is inessential in F_2. Let $D_2 \subset F_2$ be the embedded disk in F_2 that has boundury α. Since $int(D_1)$ is disjiont from F_2 then $D_1 \cup D_2$ is an embedded 2-sphere in M. Since M is irreducible, $D_1 \cup D_2$ bounds an embedded 3-ball, B.



Hence, there is an isotopy of F_1 supported in ~~B~~ a nbh of B after which $F_1 \cap D_2 = \emptyset$. This isotopy ~~decreases the~~ strictly decreases the number of components in $F_1 \cap F_2$. Hence, repeat this process untill we eliminate all components of $F_1 \cap F_2$ that are inessential in F_1 or inessential in F_2. Thus, after these isotopies $F_1 \cap F_2$ is a collection of curves that are essential in both F_1 and F_2. $\square$

Grad knot theory class

- Show that any two incompressible tori in an irreducible 3-manifold M can be isotoped to intersect in a collection of curves that is empty or has the property that each component is pairwise isotopic in each surface.
- Show that any two embedded copies of S^2 in an irreducible 3-manifold can be isotoped to be disjiont.

Def] A s.c.c. α embedded in a compact surface with boundary F is essential if it does not bound an embedded disk and is not isotopic to a component of ∂F.

If $\alpha \subset F$ is isotopic to a component of ∂F, we say α is "boundary parallel" in F.

~~Alternative def of~~

~~Given a 3-manifold~~

Given a compact 3-manifold M with boundary a properly embedded surface $F \subset M$ is essential if it is incompressible and is not isotopic to a subset of ∂M. If F is isotopic to a subset of ∂M, we say F is "boundary parallel" in M.

Alternative def/ ~~A link complement $S^3 - \eta(L)$~~

A link $K \subset S^3$ is prime if the link complement $S^3 - \eta(L)$ contains no essential meridonal properly embedded annuli.

Exercise | Suppose $K \subset S^3$ is a non-split, prime link $K \subset S^3$. Let P_1 and P_2 be two essential, properly embedded, meridianal 4-punctured spheres. Show that P_1 and P_2 can be isotoped to intersect in a collection of s.c.c. that are all pairwise isotopic in both P_1 and P_2.

Hard: What can we say if K is a hyperbolic knot?

Def | A properly embedded arc α in a compact surface with boundary is essential if it is not boundary parallel.

~~A surface F~~

Def | a compact surface F properly embedded in a compact 3-manifold w/ boundary M is boundary-compressible if there exists a disk D embedded in M s.t. $\partial D = \alpha \cup \beta$, $D \cap F = \alpha$ is an essential arc in F and $D \cap \partial M = \beta$.

Ex | Suppose M is a compact irreducible 3 manifold with boundary s.t. ∂M is a non empty collection of tori. Show if F is a properly embedded ∂-compressible surface then F is compressible or F is a ∂-parallel annulus.