Summer 2015 Knot Theory Notes

Def. of Knot

Intuitive: A knot is a loop in 3D space.

From differential topology: A knot is a smooth embedding of S^1 in $\mathbb{R}^3$.

Def. of smooth Recall from vector calc. that a parametrized curve given by $v(t) = \langle f(t), g(t), h(t) \rangle$ is smooth if each of $f(t)$, $g(t)$ and $h(t)$ are ∞-ly differentiable.

Def. of embedding A map $f: X \to Y$ is an embedding if it is continuous, one-to-one and onto its image.

Def. of S^1 $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Knot Equivalence

Intuitive: Knots K_1 and K_2 are equivalent if K_1 can be tangled, untangled, bent and stretched to coincide exactly with K_2.

From differential topology: Two knots $f: S^1 \to \mathbb{R}^3$ and $g: S^1 \to \mathbb{R}^3$ are equivalent if there exists a smooth map

$$H: S^1 \times [0,1] \longrightarrow \mathbb{R}^3 \quad \text{such that}$$

1) $H(s,t): S^1 \times \{t\} \to \mathbb{R}^3$ is a diffeomorphism for every $t \in [0,1]$.

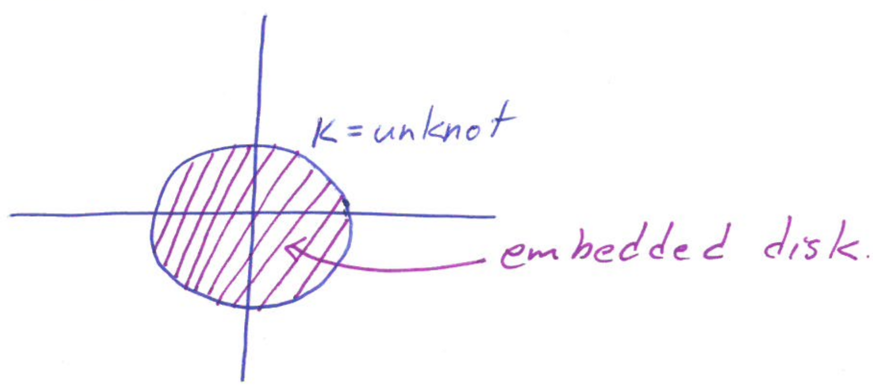
<u>and</u>

2) $H(f(x), 1) = g(x)$ for $x \in S^1$.

If two knots are equivalent in this manner we say they are <u>ambient isotopic</u>.

<u>Additional definitions</u>

- A <u>link</u> is an embedding of multiple circles $\left(\coprod_{i=1}^{n} S^1\right)$ into $\mathbb{R}^3$. Two links are equivalent if they are ambient isotopic.
- The <u>unknot</u> is any knot in $\mathbb{R}^3$ that is ~~ambient~~ ambient isotopic to the standard embedding of S^1 in the xy-plane. Equivalently, any knot that bounds an embedded disk in $\mathbb{R}^3$ is the unknot.



Knot Diagrams

2D is easier than 3D, so we want to study 2D representations of knots.

Let $p: \mathbb{R}^3 \to \mathbb{R}^2$ via $p(x,y,z) = (x,y)$ be the standard projection map onto the xy-plane.

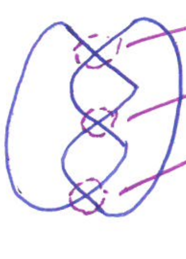
Let $K \subset \mathbb{R}^3$ be a knot (Here we are abusing notation since tecnically knots are mapps, but that is OK).

Let $P|_K : K \to \mathbb{R}^2$ be the restriction of P to the subset K.

Theimage of the map $P|_K$ is called a <u>projection</u>.

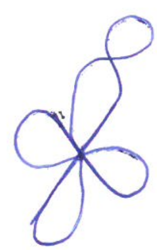
If every point in the range of $P|_K$ has at most two preimages, we say the image of $P|_K$ is a <u>regular projection</u>.

regular projection of a knot



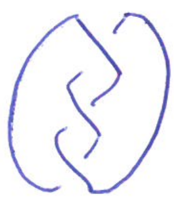
double points in a regular knot projection are called <u>crossings</u>

not a regular projection of a knot



Def. of knot diagram A knot diagram for a knot K is a regular projection of K together with lables at each crossing that indicate which strand goes over and which goes under.

diagram of the trefoil



projection

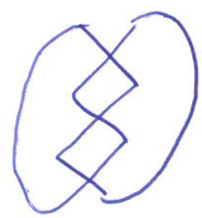
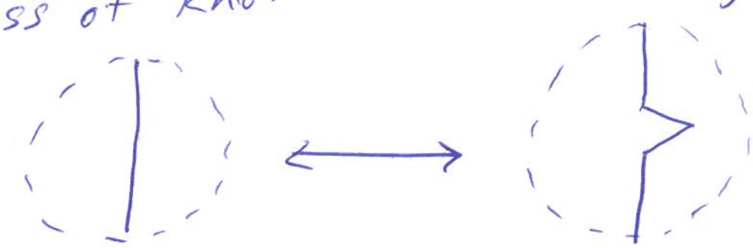
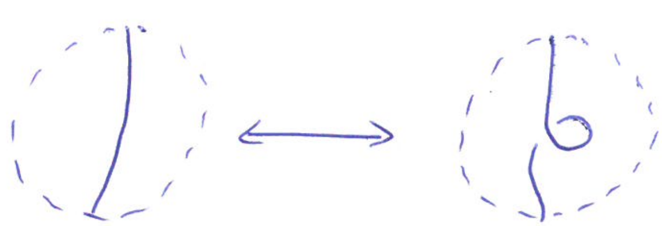


diagram of the unknot

Big Question When do two knot diagrams represent the same knot?

Reidermeister Moves Reidermeister moves are local moves that can be applied to knot diagrams to change them, but not alter the ambient isotopy class of knot that the diagram represents.

RO:  (sometimes called <u>planar isotopy</u>)

RI: 

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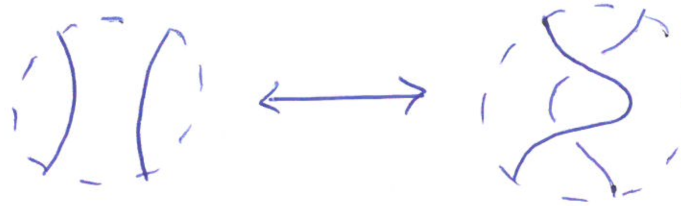
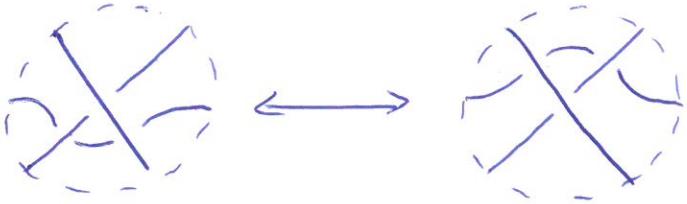
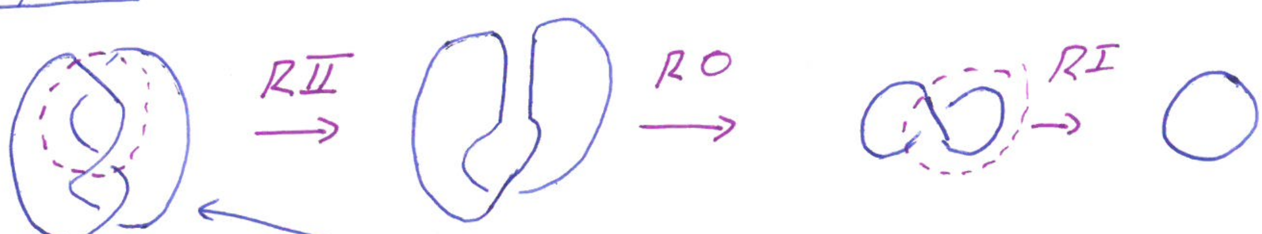


RIII :



<u>Big Th$\underline{\text{m}}$ (Reidermeister)</u> Two knots are equivalent if-and-only-if their diagrams are related by a finit sequence of Reidermeister moves.

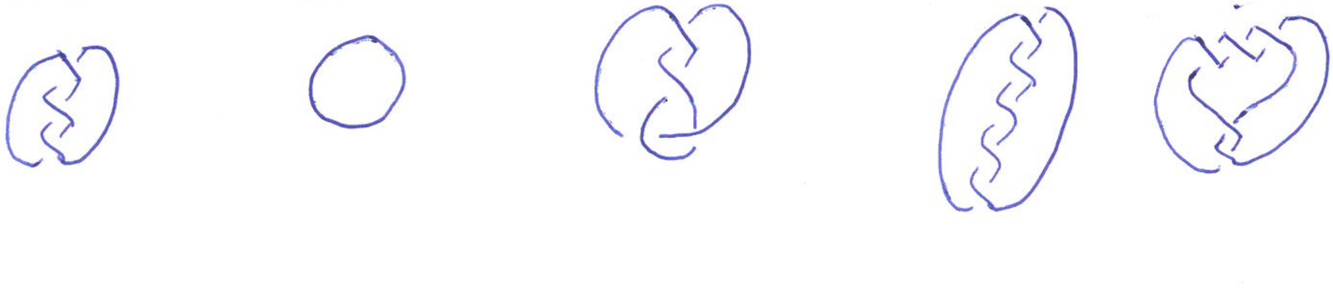
<u>Example</u>



So, the first diagram is a diagram of the unknot.

<u>Examples of knots</u>

3_1 = Trefoil unknot = 0, figure 8 = 4_1 5_1 5_2



Big Question | How do we tell knots apart?

Let $\mathcal{K}$ be the set of ambient isotopy classes of knots in $\mathbb{R}^3$.

A knot invariant is a function

$$F : \mathcal{K} \longrightarrow \text{Some Mathematical Object}$$

Example |

The crossing number of a knot K, denoted $c(K)$, is the minimal number of crossings in any diagram of K.

Hence, crossing number (i.e. $c : \mathcal{K} \to \mathbb{Z}$) is a knot invariant.

Example | The crossing number of the unknot is zero.

Exercise | Show that the crossing number of the trefoil is 3.

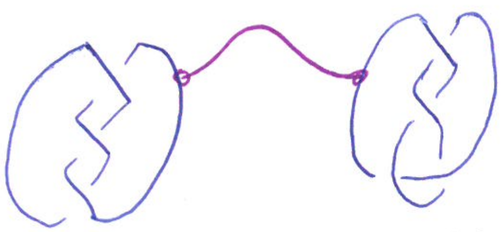
Connected Sum

In mathematics it is often helpful to have a process by which you can take two existing objects and amalgamate them together into a single object.

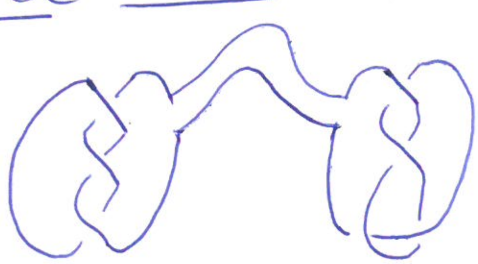
Given knots K_1 and K_2

K_1 K_2

we can draw an arc from K_1 to K_2



and do surgery along that arc to form a <u>connected</u> <u>sum</u> $K_1 \# K_2$.



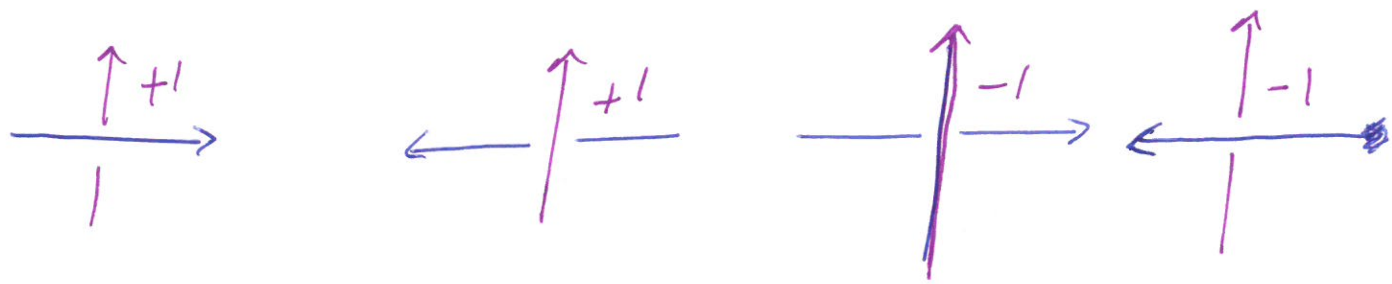
$K_1 \# K_2$.

Open Conjecture: $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

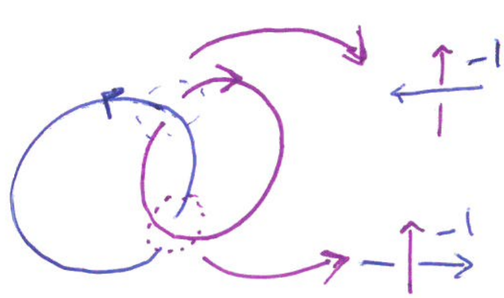
Combinatorial definition of linking number!

Given a link L with two oriented components, A and B). we can calculate the linking number of L using the following algorythm:

Given a diagram of L, we assign a $+1$ or a -1 to every crossing which is incident to <u>both</u> A and B. We then sum these integers to determine the <u>linking number of L</u>. The $+1$s and -1s are assigned in the following way:



Examples!

$L =$ 

 $= M$

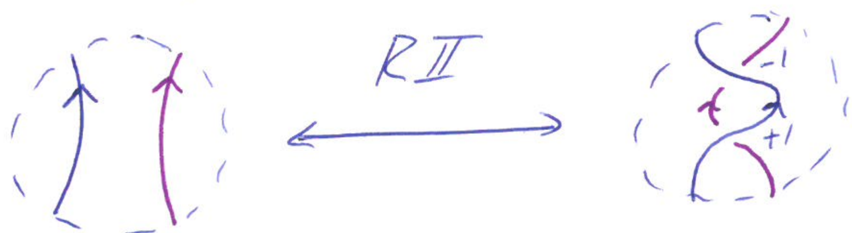
so, $lk(L) = -2$

$lk(M) = -2$

We can use Reidermeister moves to show <u>linking number</u> is a link invariant.

We need to show that every possible Reidermeister move preserves linking number.

<u>Example 1</u>



$lk = 0$ $\qquad$ $lk = +1 - 1 = 0$

So, in this <u>specific</u> configuration linking number is preserved.

Now, check <u>all</u> possible configurations. Repeat this process for all other Reidermeister moves to establish invariance.